Mathematical Methods Exam 1: Solutions

Question 1

a. $\frac{x+2}{x-3} = 1 + \frac{5}{x-3}$

$\frac{x+2}{x-3} = \frac{(x-3)+5}{x-3}$

$= \frac{(x-3)}{x-3} + \frac{5}{x-3}$ **1M**

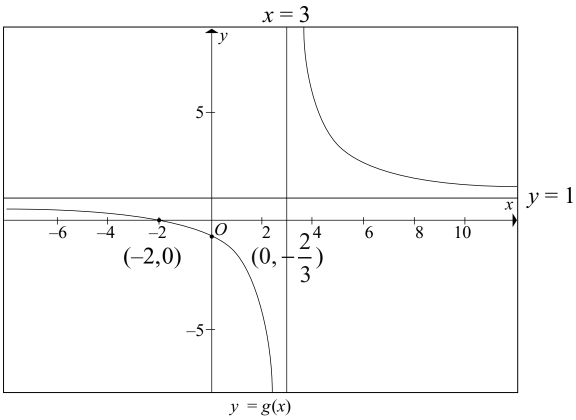
$= 1 + \frac{5}{x-3}$

Alternatively, use the long division algorithm.

$$\begin{array}{r} 1 \\ x-3 \overline{) x+2} \\ \underline{x-3} \\ 5 \end{array}$$ **1M**

$\frac{x+2}{x-3} = 1 + \frac{5}{x-3}$

b.



Shape **1A**

Asymptotes $y = 1$ and $x = 3$ **1A**

Intercepts $\left(0, -\frac{2}{3}\right)$ and (–2, 0) **1A**

c. $(-\infty, -2] \cup (3, \infty)$ **1A**

Question 2

a. $a = \frac{1}{2}$ **1A**

b. Let $y = \log_e|2x-1|$, where $x < \frac{1}{2}$

For the inverse swap x and y

$x = \log_e|2y-1|$, where $y < \frac{1}{2}$ **1A**

$$x = \begin{cases} \log_e(2y-1), & y > \frac{1}{2} \\ \log_e(1-2y), & y < \frac{1}{2} \end{cases}$$

$e^x = 1 - 2y$

$y = \frac{1-e^x}{2}$

$f^{-1}(x) = \frac{1-e^x}{2}$ **1A**

Question 3

$4^x - 5(2^x) = k$

Let $a = 2^x$, $a > 0$

$a^2 - 5a - k = 0$ **1M**

$a = \frac{5 \pm \sqrt{25+4k}}{2}$

$0 < \Delta < 25$ as $a > 0$

$0 < 25 + 4k < 25$ **1M** for discriminant
1A for restriction

$-\frac{25}{4} < k < 0$ **1A**

Question 4

a. $f(g(x)) = (1 - \log_e(2x))^{\frac{1}{3}}$ **1A**

b. By the chain rule,

$$f'(g(x)) \times g'(x) = \frac{1}{3}(1 - \log_e(2x))^{\frac{-2}{3}} \times \frac{-1}{x}$$ **1M**

Substitute $x = \frac{1}{2}$ into the derivative to find m.

$$m = \frac{1}{3}(1 - \log_e(1))^{\frac{-2}{3}} \times -2$$

$$= \frac{-2}{3}$$ **1M**

$$f\left(g\left(\frac{1}{2}\right)\right) = (1 - \log_e(1))^{\frac{1}{3}} = 1$$ **1M**

The equation of the tangent is

$$y - 1 = \frac{-2}{3}(x - \frac{1}{2})$$

$$y = -\frac{2}{3}x + \frac{4}{3}$$ **1A**

Question 5

$$\text{Area} = -\int_{-1}^{0}\left(x(x+1)^2\right)dx = \int_{0}^{-1}\left(x(x+1)^2\right)dx$$ **1A**

$$= \int_{0}^{-1}\left(x^3 + 2x^2 + x\right)dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^{-1}$$ **1M**

$$= \left((\frac{1}{4} - \frac{2}{3} + \frac{1}{2}) - 0\right)$$

$$= \frac{3 - 8 + 6}{12}$$

$$= \frac{1}{12} \text{ units}^2$$ **1A**

Question 6

a. i. Range: [–4 – 2, 4 – 2] = [–6, 2] **1A**

ii. Period $= \frac{2\pi}{\pi/6} = \frac{2\pi}{1} \times \frac{6}{\pi} = 12$ **1A**

b. Solve $4\sin\left(\frac{\pi}{6}t\right) - 2 = 0$.

$$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2}$$ **1M**

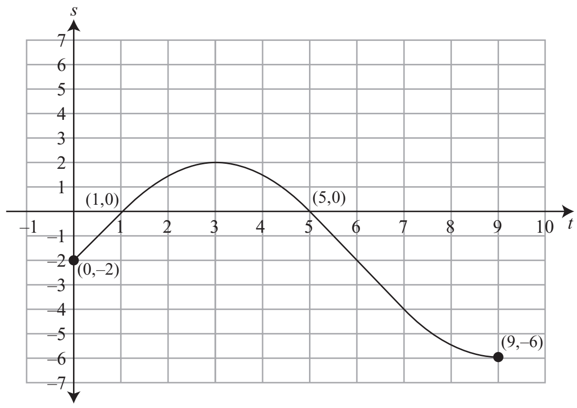
$$\frac{\pi}{6}t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$$

$$t = \frac{\pi}{6} \times \frac{6}{\pi}, \frac{5\pi}{6} \times \frac{6}{\pi}, \frac{13\pi}{6} \times \frac{6}{\pi}, \ldots$$

Since $t \in [0, 9]$,

$t = 1$ *or* $t = 5$ **1A**

c.



Correct shape: **1A**
Coordinates of x-axis intercepts labelled: **1A**
Endpoints labelled: **1A**

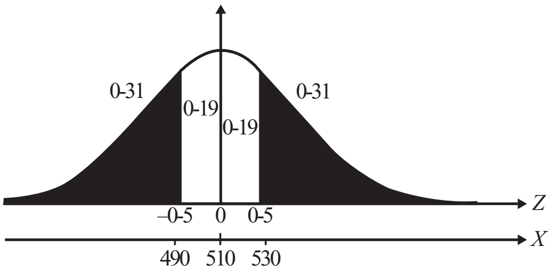
Question 7

a. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

Therefore $\Pr(430 < X < 590) \approx 0.95$

$k = 430$ **1A**

b.



$\Pr(X < 530) = 1 - \Pr(Z < -0.5)$ **1M**

$= 1 - 0.31$

$= 0.69$ **1A**

c. $\Pr\left(X < 530 | X > 510\right) = \dfrac{\Pr\left(X > 510 \cap X < 530\right)}{\Pr\left(X > 510\right)}$

$= \dfrac{\Pr\left(510 < X < 530\right)}{\Pr\left(X > 510\right)}$ **1M**

$= \dfrac{0.19}{0.5} = 0.19 \times 2$

$= 0.38$ **1A**

Question 8

a. For f to be a probability density function,

$\int_{-\infty}^{\infty} f(x) = 1$. Therefore,

$0 + a\int_{-1}^{3}(x+1)dx = 1$

$a\left[\dfrac{x^2}{2} + x\right]_{-1}^{3} = 1$ **1M**

$a\left[\left(\dfrac{9}{2} + 3\right) - \left(\dfrac{1}{2} - 1\right)\right] = 1$

$8a = 1$

$a = \dfrac{1}{8}$, *as required* **1M**

b.

$\Pr(X < 0) = \dfrac{1}{8}\int_{-1}^{0}(x+1)dx$

$= \dfrac{1}{8}\left[\dfrac{x^2}{2} + x\right]_{-1}^{0}$

$= \dfrac{1}{8}\left[0 - \left(\dfrac{1}{2} - 1\right)\right]$

$= \dfrac{1}{16}$ **1A**

c.

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= 0 + \dfrac{1}{8}\int_{-1}^{3}(x^2 + x)dx$

$= \dfrac{1}{8}\left[\dfrac{x^3}{3} + \dfrac{x^2}{2}\right]_{-1}^{3}$ **1M**

$= \dfrac{1}{8}\left[\left(9 + \dfrac{9}{2}\right) - \left(-\dfrac{1}{3} + \dfrac{1}{2}\right)\right]$

$= \dfrac{1}{8}\left[\dfrac{26}{2} + \dfrac{1}{3}\right]$

$= \dfrac{1}{8} \times \dfrac{40}{3}$

$= \dfrac{5}{3}$ **1A**

d.

$$\frac{1}{8}\int_{-1}^{m}(x+1)\,dx=\frac{1}{2}$$

$$\int_{-1}^{m}(x+1)\,dx=4$$

$$\left[\frac{x^2}{2}+x\right]_{-1}^{m}=4 \qquad \textbf{1M}$$

$$\left[\left(\frac{m^2}{2}+m\right)-\left(\frac{1}{2}-1\right)\right]=4$$

$$\frac{m^2}{2}+m+\frac{1}{2}=4$$

$$m^2+2m-7=0$$

Use quadratic formula or complete the square

$$m=\frac{-2\pm\sqrt{4+28}}{2}$$

$$=\frac{-2\pm\sqrt{32}}{2}$$

$$=\frac{-2\pm 4\sqrt{2}}{2}$$

$$=-1\pm 2\sqrt{2} \qquad \textbf{1M}$$

Note that $-1-2\sqrt{2}$ is outside the domain because $m>-1$.

Median value is $-1+2\sqrt{2}$ **1A**