THE SCHOOL FOR EXCELLENCE (TSFX)
UNIT 4 MATHEMATICAL METHODS 2006

WRITTEN EXAMINATION 2

Reading Time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOKLET

Structure of Booklet

Section	Number of questions	Number of questions to be answered	Number of marks
1	22	22	22
2	4	4	58
			Total 80

This examination has two sections: Section 1 (multiple-choice questions) and Section 2 (extended-answer questions).

You must complete both parts in the time allocated. When you have completed one part continue immediately to the other part.

Students are permitted to bring into the examination rooms: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved **graphics** calculator (memory **DOES NOT** need to be cleared) and, if desired, one scientific calculator.

Students are **NOT** permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Students are **NOT** permitted to bring mobile phones and/or any electronic communication devices into the examination room.

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SECTION 1 – MULTIPLE CHOICE QUESTIONS

Instructions for Section 1

Answer all questions in this part on the answer sheet provided for multiple-choice questions.

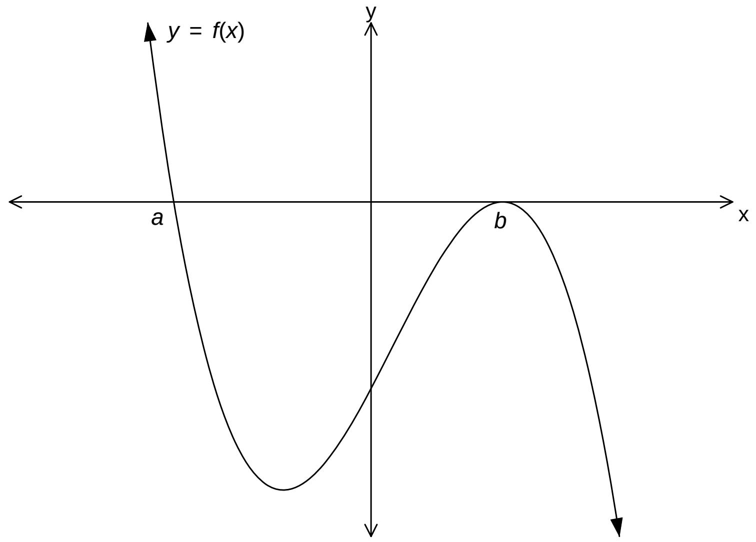
Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers. You should attempt every question.

No marks will be given if more than one answer is completed for any question.

QUESTION 1



The graph shown could be that of a function f whose rule is

A $(x-a)(x-b)^2$

B $(x+a)(x-b)^2$

C $(x+a)(b-x)^2$

D $(a-x)(x-b)^2$

E $(x-a)(b-x)^2$

QUESTION 2

The graph of the function with equation $y = |x^2 - 1|$ is reflected in the x-axis, dilated by a scale factor of 2 from the y-axis and then translated by +3 units from the x-axis.
The equation of the transformed graph is

A $y = -|4x^2 + 2|$

B $y = -\frac{1}{4}|x^2 - 4| + 3$

C $y = -\frac{1}{4}|x^2 + 2|$

D $y = -|2x^2 - 1| + 3$

E $y = -|4x^2 - 1| + 3$

QUESTION 3

The domain of the function $y = \sqrt{\frac{x-3}{x-6}}$ is

A [3, 6)

B $(6, +\infty)$

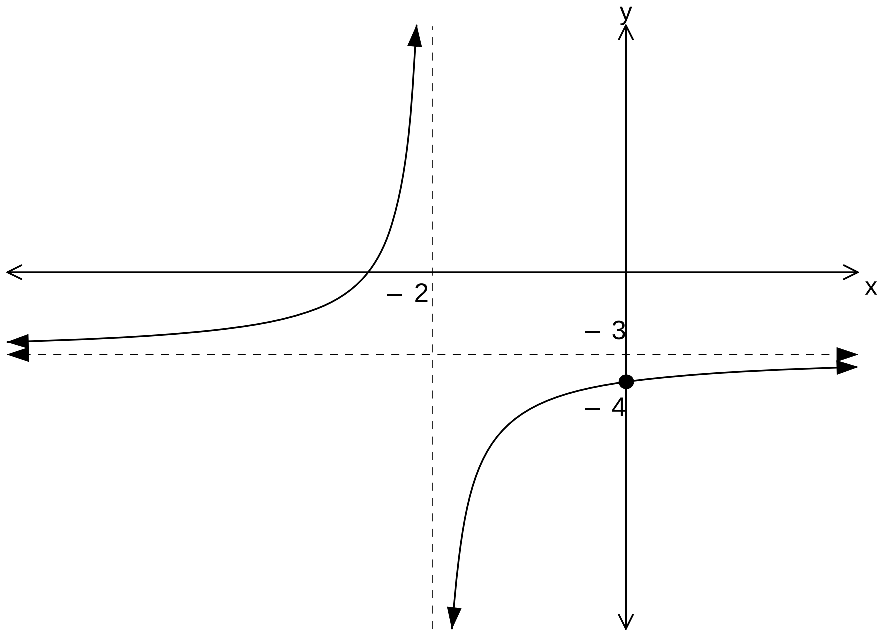
C $[3, +\infty)$

D $R \setminus \{6\}$

E $(-\infty, 3] \cup (6, +\infty)$

QUESTION 4

The graph of the function with equation $y = \dfrac{1}{Ax+B} - C$, where A, B and C are real constants, is shown below.



The values of A, B and C respectively are

A $\quad A = -\dfrac{1}{2} \qquad B = -1 \qquad C = 3$

B $\quad A = -1 \qquad B = -2 \qquad C = -3$

C $\quad A = 1 \qquad B = 1 \qquad C = 3$

D $\quad A = -2 \qquad B = 1 \qquad C = -3$

E $\quad A = 1 \qquad B = -2 \qquad C = 3$

QUESTION 5

The function $f(x)=2x^3-12x+5$ has a turning point at $\left(\sqrt{2},\ 5-8\sqrt{2}\right)$. It follows that the function $g(x)=3+f(2-x)$ has a turning point at

A $\left(2+\sqrt{2},\ 8-8\sqrt{2}\right)$

B $\left(\sqrt{2}-2,\ 8\sqrt{2}-2\right)$

C $\left(\sqrt{2}+2,\ 8\sqrt{2}-2\right)$

D $\left(2-\sqrt{2},\ 8-8\sqrt{2}\right)$

E $\left(3+\sqrt{2},\ 8\sqrt{2}-7\right)$

QUESTION 6

The function $f:(-\infty,a]\to R$ with rule $f(x)=x\sin\left(\frac{1}{x}\right)$ will have an inverse function provided

A $a=0$

B $a=-0\cdot 222555$

C $a=-0\cdot 222544$

D $a=-0\cdot 129446$

E $a=-0\cdot 159155$

QUESTION 7

Two functions *f* and *g* are defined as follows:

$$f: A \to R \text{ with rule } f(x) = \sqrt{3-2x}$$

$$g: B \to R \text{ with rule } g(x) = x^2 - \frac{5}{2}$$

The function $f(g(x))$ will be defined if

A $\quad B = \{x : x \le 2\}$

B $\quad A = \left\{x : -2 \le x \le \frac{3}{2}\right\}$

C $\quad B = \left\{x : x \le -\frac{5}{2}\right\}$

D $\quad A = \left\{x : x \le \frac{3}{2}\right\}$

E $\quad B = \{x : -2 \le x \le 2\}$

QUESTION 8

The largest set of real values of a for which $|a^2 - 2a| \ge 1$ is

A $\quad a < 1 - \sqrt{2}$ and $a > 1 + \sqrt{2}$

B $\quad a < -2$ and $a > 4$

C $\quad a \le -1$ and $a \ge 1$

D $\quad a \le -1 - \sqrt{2}$ and $a \ge -1 + \sqrt{2}$

E $\quad a < 1 - \sqrt{2}$, $a = 1$ and $a > 1 + \sqrt{2}$

QUESTION 9

The product of the first three positive solutions of the equation $2\cos\left(\frac{x}{2}\right)+\sqrt{3}=0$ is

A $\quad \frac{77\pi^3}{27}$

B $\quad \frac{595\pi^3}{27}$

C $\quad \frac{143\pi^3}{27}$

D $\quad \frac{385\pi^3}{27}$

E $\quad \frac{29\pi^3}{27}$

QUESTION 10

If $\cos^2(3b)+2\sin(3b)=-2$, where $-\frac{\pi}{2}\le b\le\frac{\pi}{2}$, then b is equal to

A $\quad -\frac{\pi}{2}$

B $\quad -\frac{\pi}{2}$ or $\frac{\pi}{6}$

C $\quad \frac{\pi}{2}$

D $\quad -\frac{\pi}{6}$ or $\frac{\pi}{2}$

E $\quad -\frac{\pi}{2}$ or $-\frac{\pi}{6}$

QUESTION 11

$\log_7\left(14^{3x}\right)$ is equal to

A $\quad 3x\left(\dfrac{\log_e(2)}{\log_e(7)}\right)$

B $\quad 3x\left(\dfrac{\log_e(7)+\log_e(2)}{\log_e(7)}\right)$

C $\quad 3x\left(1+\log_e(2)\right)$

D $\quad 3x\left(\log_7(2)\right)$

E $\quad 3x+\log_7(2)$

QUESTION 12

Let $f(x)$ and $g(x)$ be functions such that $f(0)=4$, $f'(0)=1$, $g(0)=-4$ and $g'(0)=5$.

If $h(x)=\dfrac{g(x)}{f(x)}$, then the value of $h'(0)$ is equal to

A $\quad \dfrac{3}{2}$

B 5

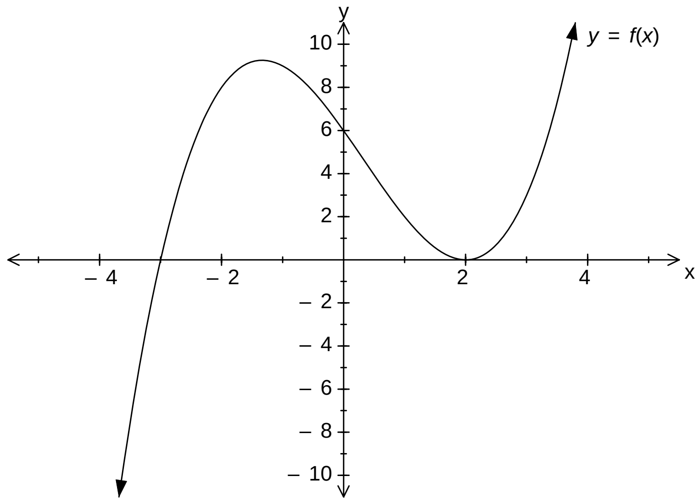
C 6

D 1

E $\quad -\dfrac{1}{2}$

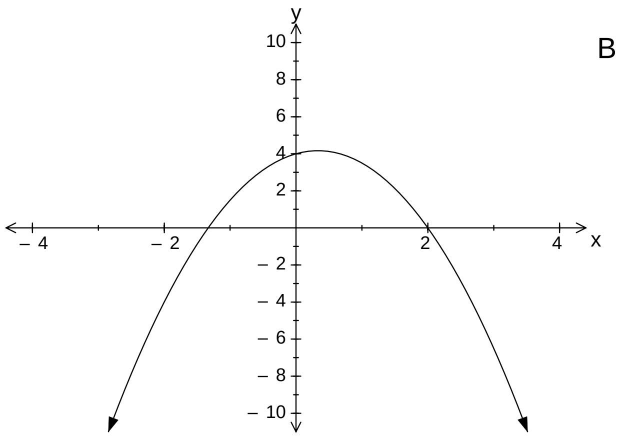
QUESTION 13

The graph of the function with equation $y = f(x)$ is shown below.

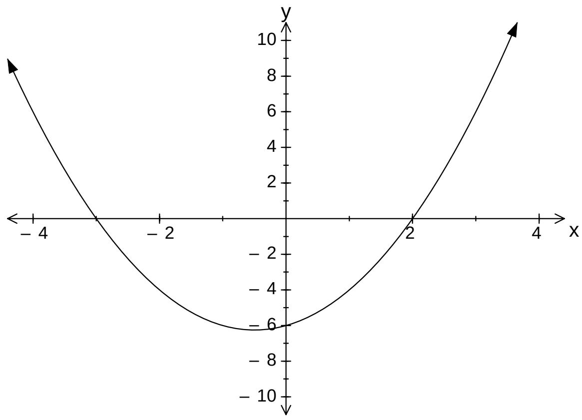


Which of the following is most likely to be the graph of the derivative function with equation $y = f'(x)$?

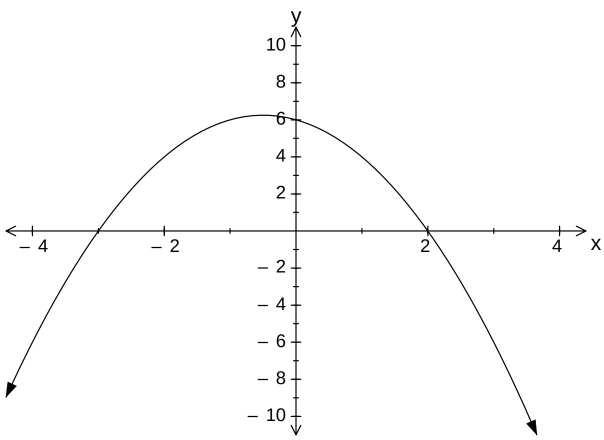
A



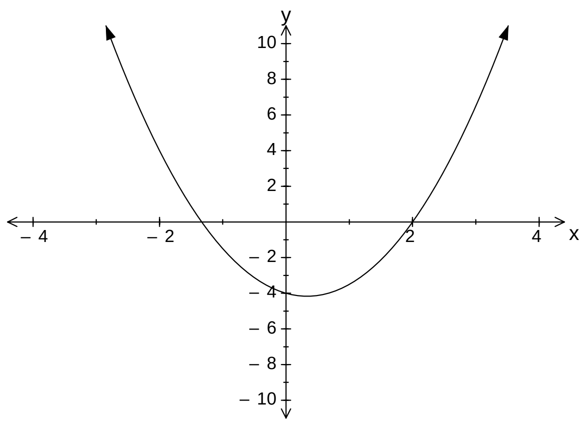
B



C



D



E None of the above.

QUESTION 14

The equation of the normal to the curve of the function with equation $y = \frac{2}{\sqrt{x}} - 2$ at the point where $x = 4$ is

A $8y + x + 4 = 0$

B $y - 8x + 33 = 0$

C $y + 8x - 31 = 0$

D $y + 2x - 7 = 0$

E $y - 8x + 31 = 0$

QUESTION 15

Let $g(x)$ be a function such that $g(1) = 4$ and $g'(1) = 9$. If $f(x) = \sqrt{g(x)}$, then the value of $f'(1)$ is equal to

A $\frac{9}{4}$

B 2

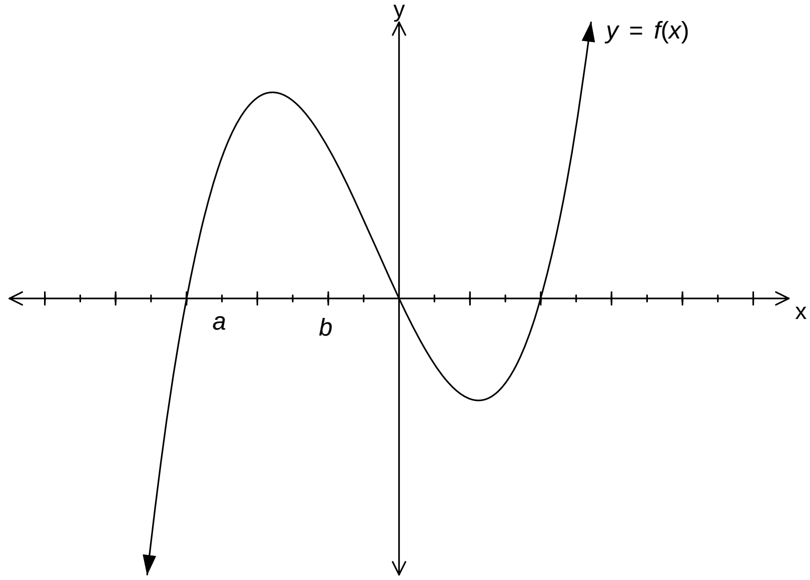
C $\frac{1}{2}$

D $\frac{1}{6}$

E 3

QUESTION 16

The graph of the function with equation $y = f(x)$ is shown below.



Let g be a function such that $g'(x) = f(x)$. On the interval (a, b), the function g will

A Have a local maximum value.

B Have a local minimum value.

C Have a zero gradient.

D Be a decreasing function.

E Be an increasing function.

QUESTION 17

If $y = \left|\cos\left(\frac{x}{2}\right)\right| - |x|$ and $-3\pi < k < -\pi$, then the rate of change of y with respect to x at $x = k$ is

A $-\frac{1}{2}\sin\left(\frac{k}{2}\right) - 1$

B $\frac{1}{2}\sin\left(\frac{k}{2}\right) + 1$

C $-\frac{1}{2}\sin\left(\frac{k}{2}\right) + 1$

D $\frac{1}{2}\sin\left(\frac{k}{2}\right) - 1$

E $2\sin\left(\frac{k}{2}\right) - 1$

QUESTION 18

If $\int_a^b g(x)\,dx = -1$ then $\int_b^a (3g(x) - 5)\,dx$ is equal to

A $-3 - 5(a - b)$

B $-3 - 5(b - a)$

C $3 + 5(b - a)$

D $3 - 5(b - a)$

E $-3 + 5(b - a)$

QUESTION 19

The random variable *X* has a normal distribution with mean 12 and variance 4. If Z has the standard normal distribution, then the probability that the value of *X* lies between 10 and 16 is equal to

A $1 - \Pr(Z > \frac{1}{2}) - \Pr(Z > 1)$

B $1 + \Pr(Z > 1) - \Pr(Z > 2)$

C $\Pr(Z < 2) + \Pr(Z > -1)$

D $1 - \Pr(Z > 1) - \Pr(Z > 2)$

E $\Pr(Z < 1) + \Pr(Z > -\frac{1}{2})$

QUESTION 20

The continuous random variable *X* has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

The median value of X, correct to four decimal places, is equal to

A $1 \cdot 0000$

B $1 \cdot 5747$

C $1 \cdot 4444$

D $0 \cdot 5969$

E $2 \cdot 0000$

QUESTION 21

A curtain rod manufacturing company used three machines on a particular day to produce curtain rods. At the end of the day the following data was given to the production manager:

	Total number of curtain rods produced	Number of defective curtain rods produced
Machine A	450	14
Machine B	350	18
Machine C	200	12

A curtain rod was selected at random from **all** of the curtain rods produced on this day by machines A, B and C and found to be **not** defective.

The probability that the curtain rod was produced by machine B is equal to

A $\frac{7}{20}$

B $\frac{9}{22}$

C $\frac{166}{175}$

D $\frac{83}{239}$

E $\frac{83}{250}$

QUESTION 22

An archer is shooting arrows at a target. Her probability of hitting the bullseye of the target is $0\cdot 4$. The minimum number of shots needed by the archer so that the probability of her hitting the bullseye at least five times is greater than $0\cdot 9$ is equal to

A 17

B 18

C 19

D 20

E 21

SECTION 2 – EXTENDED ANSWER QUESTIONS

Instructions For Section 2

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or anti-derivative.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

QUESTION 1

In the hidden depths of the Amazon jungle in Peru there lives a secret Incan tribe. This tribe is guardian to a legendary diamond statue, which it keeps in a stone chest that sits on a rock ledge on the banks of the piranha infested Amazon river. The rock ledge is $6 \cdot 4$ metres above the bottom of the river. Galaxian Highscore, the intrepid adventurer, is keen to possess this diamond statue. Because the piranha fish eat human flesh, the statue can only be safely taken from the stone chest when the chest is completely uncovered by water.

Galaxian knows that in this part of the jungle the height of the water above the bottom of the Amazon River can be modelled by the function:

$$d = 10 + 4\cos\left(\frac{\pi}{14}t\right)$$

where d is height (in metres) above the bottom of the river at time t hours after 12:00 noon on any given Monday.

a. Explain why the height of the water is the same at 12:00 noon on every Monday.

2 marks

b. (i) Determine, correct to the nearest minute, the first time after 12:00 noon on Monday when the stone chest is completely uncovered by water.

2 marks

(ii) Determine, correct to the nearest minute, the amount of time that Galaxian Highscore has to safely take the statue from the stone chest.

1 mark

Unfortunately for Galaxian Highscore, the Incan tribe captures her before she is able to get the diamond statue. They tie her by a vine 20 metres above the bottom of the river at midnight on Monday and slowly lower her at a constant rate such that she will first enter the piranha infested water **after** midnight on Tuesday.

c. (i) Show that the equation of the line that is tangent to the curve $d = 10 + 4\cos\left(\frac{\pi}{14}t\right)$ and passes through the point (12, 20) is given by:

$d = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)(t-12)+20$, where t_1 is the t-coordinate of the tangent point.

2 marks

(ii) **Hence** find the value of t_1, correct to three decimal places.

2 marks

(iii) **Hence** find the maximum possible rate at which Galaxian is being lowered towards the river, correct to three decimal places.

1 mark

The Incan tribe decides to release Galaxian if she can solve the ancient Incan puzzle of finding solutions to $\sin(3x) = \cos\left(x + \frac{\pi}{4}\right)$.

d. Find in exact form the real solutions to the equation $\sin(3x) = \cos\left(x + \frac{\pi}{4}\right)$ which satisfy $0 \le x \le \pi$.

3 marks

Total Marks = 13

QUESTION 2

The Lambert W function is defined to be the inverse of xe^x. It follows from the definition of the Lambert W function that if $a = xe^x$, then $x = \mathrm{W}(a)$.

a. (i) State the solution to the equation $xe^x = 6$ in terms of the Lambert W function.

1 mark

(ii) Find, correct to four decimal places, the value of $\mathrm{W}(3)$.

1 mark

(iii) Show that the solution to the equation $xe^{3x} = 2$ is $x = \dfrac{\mathrm{W}(6)}{3}$.

2 marks

b. (i) Show that the equation $x + e^{x} = 2$ can be written in the form $e^{-x}(2 - x) = 1$.

1 mark

(ii) Let $2 - x = t$ in the above equation. Show that $te^{t} = e^{2}$.

1 mark

(iii) **Hence** show that the solution to the equation $x + e^{x} = 2$ is $x = 2 - \mathrm{W}(e^{2})$.

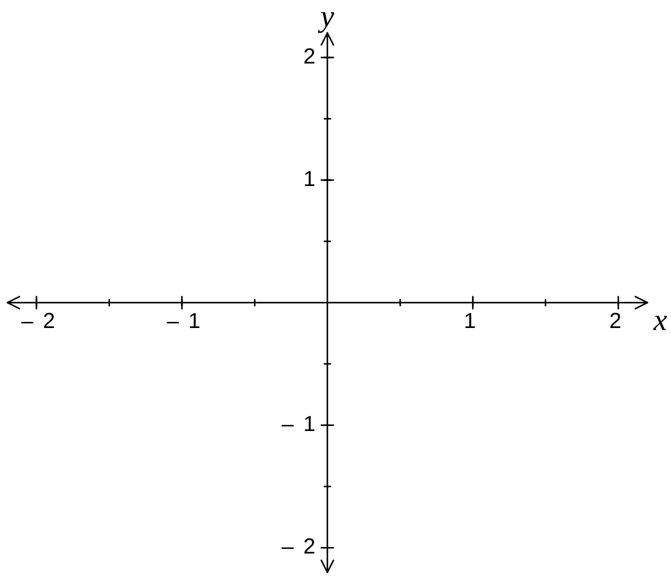
1 mark

c. (i) Find the exact largest value of a such that the function

$f:(-\infty, a] \to R$ with rule $f(x) = xe^{-x}$ has an inverse function f^{-1}.

2 marks

(ii) For the value of a found above, sketch the graph of $y = f^{-1}(x)$ on the set of axes below. Label any end-point with its coordinates. Label any axis intercept with its coordinates.



3 marks

(iii) Determine the rule for f^{-1} in terms of the Lambert W function.

2 marks

(iv) **Hence** find the exact value of $\mathrm{W}\left(-\frac{1}{e}\right)$.

2 marks

Total = 16 marks

QUESTION 3

a. (i) Use calculus to find the derivative of $(x^2+\alpha x+\beta)\sqrt{2x-3}$.

2 marks

(ii) **Hence** express the derivative of $(x^2+\alpha x+\beta)\sqrt{2x-3}$ in the form $\dfrac{q(x)}{\sqrt{2x-3}}$,

where $q(x)$ is a quadratic expression.

2 marks

b. **Hence** find an anti-derivative of $\dfrac{2x(5x-9)}{\sqrt{2x-3}}$.

3 marks

c. (i) **Hence** use calculus to find an expression that gives the exact area between the curve $y = \frac{2x(5x-9)}{\sqrt{2x-3}}$ and the lines $x = \frac{8}{5}$, $x = 2$ and $y = 0$.

2 marks

(ii) Find, correct to three decimal places, the area between the curve $y = \frac{2x(5x-9)}{\sqrt{2x-3}}$ and the lines $x = \frac{8}{5}$, $x = 2$ and $y = 0$.

1 mark

d. Find all exact values of α for which $f(x)=(x^2+\alpha x-1)\sqrt{2x-3}$ is always an increasing function.

4 marks

Total = 14 marks

QUESTION 4

A continuous random variable X has a Weibull distribution if it has a probability distribution function given by:

$$f(x) = \begin{cases} \dfrac{\alpha}{\beta} x^{\alpha-1} e^{-x^{\alpha}/\beta} & \text{if} \quad x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where α and β are positive constants. The Weibull distribution provides a good model for the length of life of many electronic components and biological plants and animals.

It has been found that the length of life (measured in thousands of hours) of an aircraft guidance system follows a Weibull distribution with $\alpha = 2$ and $\beta = 10$.

a. Find correct to the nearest hour:

(i) The expected value of the length of life of an aircraft guidance system.

2 marks

(ii) The standard deviation of the length of life of an aircraft guidance system.

3 marks

b. Find correct to four decimal places the probability that a randomly selected aircraft guidance system will have a length of life that exceeds four thousand hours.

2 marks

c. If three guidance systems are used independently, find correct to four decimal places the probability that exactly one of the three fails prior to four thousand hours of use.

2 marks

d. A particular aircraft guidance system is known to have already lasted for four thousand hours. Find correct to four decimal places the probability that the guidance system will last for less than five thousand hours.

3 marks

e. Find correct to the nearest hour the expected length of life of an aircraft guidance system that has already lasted for four thousand hours.

4 marks

Total Marks = 16

BONUS QUESTION

QUESTION 1

Let $y = f(g(x))$, where:

$$f(-1) = 10\,, \qquad f'(-1) = -1$$

$$g(2) = -1\,, \qquad g'(2) = -4$$

a. Find the value of y when $x = 2$.

1 mark

b. Find the value of $\frac{dy}{dx}$ when $x = 2$.

2 marks

Now let $f(x) = 2 + 4\sqrt{3 - x}$ and $h(x) = 1 - \log_e(2x - 1)$.

c. (i) Find the exact maximal domain on which $y = f(h(x))$ is defined.

3 marks

(ii) Find the rule for $y = f(h(x))$.

1 mark

(iii) If $f(h(\beta)) = 10$, find the exact value of β.

2 marks

Total Marks = 8