THE SCHOOL FOR EXCELLENCE

UNIT 4 MATHEMATICAL METHODS 2006
COMPLIMENTARY WRITTEN EXAMINATION 2 - SOLUTIONS

SECTION 1 – MULTIPLE CHOICE QUESTIONS

QUESTION 1 Answer is D

The function has an *x*-intercept at $x = a$ and an *x*-intercept that is also a turning point at $x = b$.

Model for rule of function: $y = A(x-a)(x-b)^2$.

Since $y \to -\infty$ as $x \to +\infty$, *A* is negative. Let $A = -1$.

Therefore: $y = -(x-a)(x-b)^2 = (a-x)(x-b)^2$.

QUESTION 2 Answer is B

$$|x^2 - 1| \to -|x^2 - 1| \to -\left|\left(\tfrac{1}{2}x\right)^2 - 1\right| = -\left|\tfrac{1}{4}x^2 - 1\right| = -\tfrac{1}{4}\left|x^2 - 4\right| \to -\tfrac{1}{4}\left|x^2 - 4\right| + 3.$$

QUESTION 3 Answer is E

It is required that $\dfrac{x-3}{x-6} \ge 0$, where $x - 6 \ne 0 \Rightarrow x \ne 6$.
Therefore either:

- $x - 3 \ge 0$ AND $x - 6 > 0 \Rightarrow x > 6$, or
- $x - 3 \le 0$ AND $x - 6 < 0 \Rightarrow x \le 3$.

QUESTION 4 Answer is A

Horizontal asymptote at $y = -3 \therefore -C = -3 \Rightarrow C = 3$.

Vertical asymptote at $x = -2 \therefore A(-2) + B = 0 \Rightarrow B = 2A$.

$(0, -4)$ is a point on the curve $\therefore -4 = \dfrac{1}{0 + B} - 3 \Rightarrow B = -1$.

Substitute $B = -1$ into $B = 2A$: $-1 = 2A \therefore A = -\dfrac{1}{2}$.

QUESTION 5 Answer is D

$g(x) = f(-(x-2)) + 3$.

Therefore: $f(x) \to f(-x) \to f(-(x-2)) \to f(-(x-2)) + 3$

Therefore: $\left(\sqrt{2},\ 5-8\sqrt{2}\right) \to \left(-\sqrt{2},\ 5-8\sqrt{2}\right) \to \left(-(\sqrt{2}-2),\ 5-8\sqrt{2}\right) \equiv \left(2-\sqrt{2},\ 5-8\sqrt{2}\right)$
$\to \left(2-\sqrt{2},\ 5-8\sqrt{2}+3\right)$.

QUESTION 6 Answer is B

f will have an inverse function if it is a 1-to-1 function.

Use the graphics calculator to draw a graph of f. The smallest value of x at which f has a turning point is $x \approx -0 \cdot 2225482$.

Therefore, f is a 1-to-1 function over the domain $x < -0 \cdot 2225482$.

QUESTION 7 Answer is E

Require $ran\ g \subseteq dom\ f$.

$$\text{dom } f = \left\{ x : 3 - 2x \geq 0 \Rightarrow x \leq \frac{3}{2} \right\}.$$

It is therefore required that:

$$x^2 - \frac{5}{2} \leq \frac{3}{2}$$

$$\begin{aligned} &\therefore x^2 \leq 4 \\ &\therefore -2 \leq x \leq 2 \end{aligned}.$$

QUESTION 8 Answer is E

By definition: $|a^2 - 2a| = \begin{cases} a^2 - 2a & \text{for } a < 0 \text{ or } a > 2 \\ 2a - a^2 & \text{for } 0 \leq a \leq 2 \end{cases}$

$a^2 - 2a = 1 \Rightarrow a^2 - 2a - 1 = 0 \Rightarrow a = 1 \pm \sqrt{2}$.

$2a - a^2 = 1 \Rightarrow a^2 - 2a + 1 = 0 \Rightarrow a = 1$.

QUESTION 9 Answer is B

$$2\cos\left(\frac{x}{2}\right)+\sqrt{3}=0 \Rightarrow \cos\left(\frac{x}{2}\right)=-\frac{\sqrt{3}}{2}$$

$\therefore \frac{x}{2}=\frac{5\pi}{6}+2n\pi$ or $\frac{x}{2}=\frac{7\pi}{6}+2n\pi$, $n\in J$.

$\therefore x=\frac{5\pi}{3}+4n\pi$ or $x=\frac{7\pi}{3}+4n\pi$.

First three positive solutions: $x=\frac{5\pi}{3},\ \frac{7\pi}{3},\ \frac{5\pi}{3}+4\pi=\frac{17\pi}{3}$.

Product of the first three positive solutions: $x=\left(\frac{5\pi}{3}\right)\left(\frac{7\pi}{3}\right)\left(\frac{17\pi}{3}\right)$.

QUESTION 10 Answer is D

$$\cos^2(3b)+2\sin(3b)=-2$$

$$\therefore [1-\sin^2(3b)]+2\sin(3b)+2=0$$

$$\therefore \sin^2(3b)-2\sin(3b)-3=0$$

$$\therefore (\sin(3b)-3)(\sin(3b)+1)=0.$$

Therefore either:

- $\sin(3b)-3=0$ (no real solution), or
- $\sin(3b)+1=0 \Rightarrow \sin(3b)=-1$

  $3b=-\frac{\pi}{2}+2n\pi$, $n\in J$

  $b=-\frac{\pi}{6}+\frac{2n\pi}{3}$.

Apply the restriction $-\frac{\pi}{2}\le b\le\frac{\pi}{2}$: $b=-\frac{\pi}{6},\ -\frac{\pi}{6}+\frac{2n\pi}{3}=\frac{\pi}{2}$.

QUESTION 11 Answer is B

$$\log_7\left(14^{3x}\right)=3x\log_7(14)=3x\log_7(2\times 7)=3x\{\log_7(2)+\log_7(7)\}=3x\{\log_7(2)+1\}.$$

Apply the change of base formula: $\log_7(2)=\frac{\log_e(2)}{\log_e(7)}$.

Therefore: $\log_7\left(14^{3x}\right)=\left(\frac{\log_e(2)}{\log_e(7)}+1\right)=\left(\frac{\log_e(2)}{\log_e(7)}+\frac{\log_e(7)}{\log_e(7)}\right)=\left(\frac{\log_e(2)+\log_e(7)}{\log_e(7)}\right)$.

QUESTION 12 Answer is A

Quotient rule: $h'(x) = \dfrac{g'(x)f(x) - f'(x)g(x)}{[f(x)]^2}$

$\therefore h'(0) = \dfrac{g'(0)f(0) - f'(0)g(0)}{[f(0)]^2} = \dfrac{(5)(4)-(1)(-4)}{[4]^2} = \dfrac{24}{16} = \dfrac{3}{2}$.

QUESTION 13 Answer is D

$y = f(x)$ has turning points at $x = 2$ and $x \approx -1\cdot 2$. Therefore $y = f'(x) = 0$ at $x = 2$ and $x \approx -1\cdot 2$.

$y = f(x)$ is an increasing function for $x > 2$. Therefore $y = f'(x) > 0$ for $x > 2$.

$y = f(x)$ has the appearance of a cubic function therefore $y = f'(x)$ has the appearance of a quadratic function.

QUESTION 14 Answer is B

To get the equation of the normal it is necessary to know the gradient of the normal and the coordinates of a point on the normal. The model $y - y_1 = m(x - x_1)$ can then be used.

Gradient of the normal when $x = 4$: $y = \dfrac{2}{\sqrt{x}} - 2 = 2x^{-1/2} - 2 \therefore \dfrac{dy}{dx} = -x^{-3/2} = -\dfrac{1}{x\sqrt{x}}$.

Therefore $m_{\text{tangent}} = -\dfrac{1}{4\sqrt{4}} = -\dfrac{1}{8}$.

Apply $(m_{\text{tangent}})(m_{\text{normal}}) = -1$: $m_{\text{normal}} = 8$.

Coordinates of a point on the normal: $x = 4 \Rightarrow y = \dfrac{2}{\sqrt{4}} - 2 = -1$.

Equation of the normal at the point where $x = 4$:

$y - (-1) = 8(x - 4) \Rightarrow y + 1 = 8x - 32 \Rightarrow y = 8x - 33$.

QUESTION 15 Answer is A

Let $y = f(x)$ and use the chain rule:

Let $w = g(x)$. Then $y = \sqrt{w} = w^{1/2}$.

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dw} \times \dfrac{dw}{dx} = \dfrac{1}{2}w^{-1/2} \times g'(x) = \dfrac{1}{2}\dfrac{1}{\sqrt{w}} \times g'(x) = \dfrac{1}{2}\dfrac{1}{\sqrt{g(x)}} \times g'(x)$.

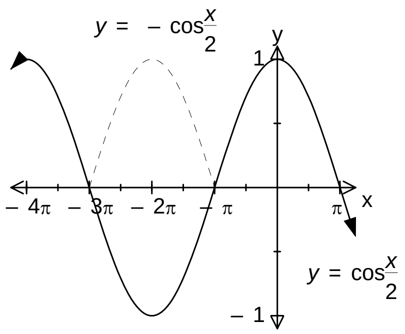
When $x = 1$: $f'(1) = \dfrac{1}{2}\dfrac{1}{\sqrt{g(1)}} \times g'(1) = \dfrac{1}{2}\dfrac{1}{\sqrt{4}} \times 9 = \dfrac{9}{4}$.

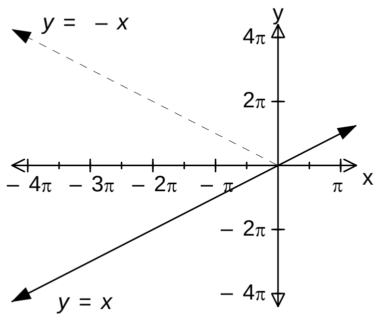
QUESTION 16 Answer is E

$y = f(x) = g'(x) > 0$ on the interval $(a,\ b)$. By definition *g* will therefore be an increasing function.

QUESTION 17 Answer is B

For $-3\pi < x < -\pi$, $\left|\cos\left(\frac{x}{2}\right)\right| = -\cos\left(\frac{x}{2}\right)$ and $|x| = -x$:





Therefore $y = -\cos\left(\frac{x}{2}\right) - (-x) = -\cos\left(\frac{x}{2}\right) + x$ over $-3\pi < x < -\pi$.

Therefore : $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{x}{2}\right) + 1$ over $-3\pi < x < -\pi$.

QUESTION 18 Answer is C

Reverse the integral terminals:

$$\int_b^a (3g(x) - 5)\,dx = -\int_a^b (3g(x) - 5)\,dx$$

$$= -3\int_a^b g(x)\,dx + \int_a^b 5\,dx = (-3)(-1) + 5(b - a)$$

$$= 3 + 5(b - a).$$

QUESTION 19 Answer is D

Variance $= \sigma^2 = 4$.

Therefore : Standard deviation $= \sigma = 2$.

Transformation formula from normal to standard normal: $Z = \dfrac{X - \mu}{\sigma}$.

Therefore: $X = 10 \Rightarrow Z = \dfrac{10-12}{2} = -1$ and $X = 16 \Rightarrow Z = \dfrac{16-12}{2} = 2$.

Therefore: $\Pr(10 < X < 16) = \Pr(-1 < Z < 2) = 1 - \Pr(Z < -1) - \Pr(Z > 2)$.

$\Pr(Z < -1) = \Pr(Z > 1)$ by the symmetry of the normal distribution.

Therefore: $\Pr(10 < X < 16) = 1 - \Pr(Z > 1) - \Pr(Z > 2)$.

QUESTION 20 Answer is B

Let the median value of X be equal to k, where $0 \le k \le 2$.

By definition:

$$\int_0^k \frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6}\,dx = \frac{1}{2}$$

$$\therefore \left[\frac{1}{6}x^3 - \frac{1}{6}x^2 + \frac{1}{6}x\right]_0^k = \frac{1}{2}$$

$$\therefore \frac{1}{6}k^3 - \frac{1}{6}k^2 + \frac{1}{6}k = \frac{1}{2}$$

$$\therefore k^3 - k^2 + k - 3 = 0$$

$\therefore k = 1 \cdot 5747$, correct to four decimal places.

QUESTION 21 Answer is D

The table can be annotated in the following way:

	Total number	Number defective	Number not-defective
Machine A	450	14	436
Machine B	350	18	332
Machine C	200	12	188
	1000	44	956

Pr(Machine B | not defective) $= \frac{332}{956} = \frac{83}{239}$.

QUESTION 22 Answer is B

Let *X* denote the random variable *number of shots that hit the bullseye*.

Then *X* ~ Binomial($p = 0 \cdot 4$, $n = ?$).

The smallest value of *n* such that $\Pr(X \geq 5) > 0 \cdot 9$ is required:

$\Pr(X \geq 5) > 0 \cdot 9 \Rightarrow 1 - \Pr(X \leq 4) > 0 \cdot 9 \Rightarrow \Pr(X \leq 4) < 0 \cdot 1$.

The most efficient approach to solving this inequality is to use the graphics calculator:

1. Define Y1= binomcdf(X, 0.4, 4). Note: In this rule X represents the sample size variable, NOT the random variable.
2. Scroll down TABLE until the first integer value of X satisfying Y1 < 0.1 is found.
3. X = 18.

SECTION 2 – EXTENDED ANSWER QUESTIONS

QUESTION 1

a. Period = $\dfrac{2\pi}{\frac{\pi}{14}} = 28$ hours.

12.00 noon Monday to 12.00 noon Monday = 7 days = $7 \times 24 = 168$ hours.

168 = 6×28 = 6 periods and hence height of water will be at the beginning of the cycle.

Alternatively: $t = 0 \Rightarrow d = 14$.

$$t = 168 \Rightarrow d = 10 + 4\cos\left(\frac{\pi}{14} \times 168\right) = 10 + 4\cos(12\pi) = 10 + 4(1) = 14.$$

b. (i) The value of t when $d = 6 \cdot 4$ is required: $6 \cdot 4 = 10 + 4\cos\left(\dfrac{\pi}{14}t\right)$.

Since only an approximate solution is required, the graphics calculator can be used:

Draw the graphs of Y1 = $10 + 4\cos(\pi\,\mathrm{X}/14)$ and Y2 = $6 \cdot 4$.

The X-coordinate of the intersection point of Y1 and Y2 gives the required value of *t*.

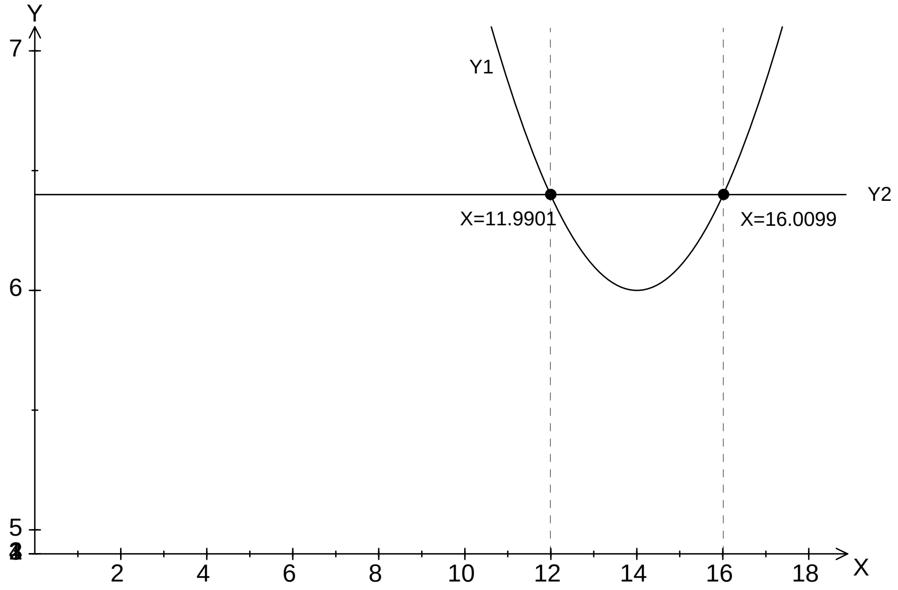
$t = 11 \cdot 9901$, which corresponds to a ‘clock time’ of 11:59 pm Monday, correct to the nearest minute.

(ii) The time interval over which $d < 6 \cdot 4$ is required.

Since only an approximate solution is required, the graphics calculator can be used:

Draw the graphs of Y1 = $10 + 4\cos(\pi\,\mathrm{X}/14)$ and Y2 = $6 \cdot 4$.
Get the X-coordinate of the intersection points of Y1 and Y2.

The first two intersection points occur when $X = 11 \cdot 9901$ and $X = 16 \cdot 0099$.



Therefore the time interval is equal to $16 \cdot 0099 - 11 \cdot 9901 = 4 \cdot 0198$ hours
= 4 hours 1 minute,

correct to the nearest minute.

c. (i) Let the tangent to $d = 10 + 4\cos\left(\frac{\pi}{14}t\right)$ be at the point (t_1, d_1) and have gradient *m*.

Then $m = d'(t_1) = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)$.

Since the tangent passes through the point $(12, 20)$ and has gradient

$m = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)$, an equation of the tangent is given by

$$d - 20 = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)(t-12)$$

$$\therefore d = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)(t-12) + 20\,.$$

(ii) Since the tangent is at the point (t_1, d_1), this point is common to both

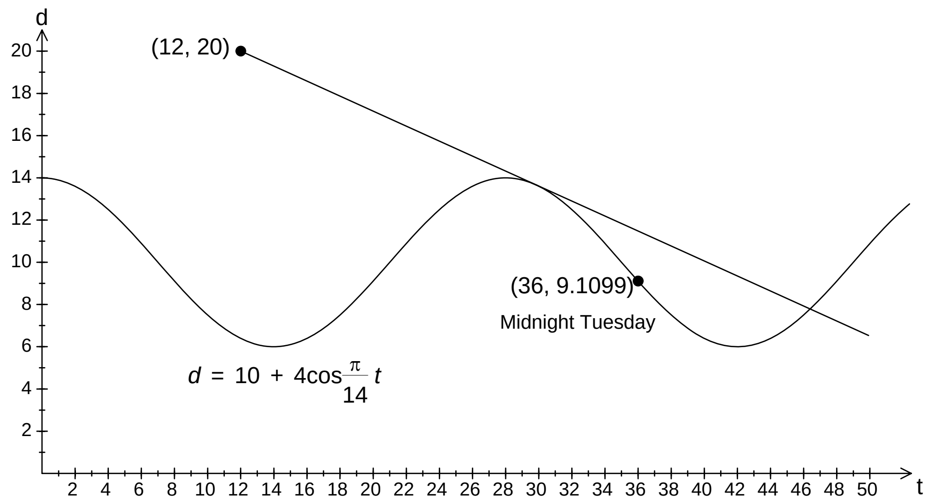
$$d = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)(t-12)+20 \text{ and } d = 10+4\cos\left(\frac{\pi}{14}t\right).$$

Therefore: $d_1 = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)(t_1-12)+20$. (1)

$d_1 = 10+4\cos\left(\frac{\pi}{14}t_1\right)$. (2)

The simultaneous solution to equations (1) and (2) for t_1 is given by the solution to $-\frac{2\pi}{7}\sin\left(\frac{\pi}{14}t_1\right)(t_1-12)+20 = 10+4\cos\left(\frac{\pi}{14}t_1\right)$ and can be found using the graphics calculator: $t_1 = 29\cdot 813$, correct to three decimal places.

(iii) If Galaxian is to first enter the water **after** midnight on Tuesday, the required maximum possible rate of change of the height of Galaxian above the water with respect to time is equal to the gradient of the tangent to $d = 10+4\cos\left(\frac{\pi}{14}t\right)$ that passes through the point $(12, 20)$:



$$m = d'(29\cdot 8129) = -\frac{2\pi}{7}\sin\left(\frac{\pi}{14}\times 29\cdot 8129\right) \approx -0\cdot 355.$$

Alternatively, the graphics calculator can be used to get the value of the derivative of $d = 10 + 4\cos\left(\frac{\pi}{14}t\right)$ at $t = 29 \cdot 8129$.

Note: In order to avoid accumulation of rounding error, a greater degree of accuracy than required in the final answer is used during the calculation.

The maximum possible rate at which Galaxian is being lowered towards the river is therefore equal to $0 \cdot 355$ metres per hour, correct to three decimal places.

d. $\sin(3x) = \cos\left(x + \frac{\pi}{4}\right)$

Write $\sin(3x)$ in terms of $\cos$ using the identity $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$.

$\therefore \sin(3x) = \cos\left(\frac{\pi}{2} - 3x\right)$

$\therefore \cos\left(\frac{\pi}{2} - 3x\right) = \cos\left(x + \frac{\pi}{4}\right)$ or Using the identity $\cos(\theta) = \cos(-\theta)$

$\therefore \frac{\pi}{2} - 3x = x + \frac{\pi}{4} + 2n\pi$ or $-\left(\frac{\pi}{2} - 3x\right) = x + \frac{\pi}{4} + 2n\pi$ where $n \in J$

$\therefore x = \frac{\pi}{16} - \frac{n\pi}{2}$ $\qquad$ $\therefore x = \frac{3\pi}{8} + n\pi$

.

Apply the restriction $0 \le x \le \pi$:

$x = \frac{\pi}{16}, \quad \frac{\pi}{16} + \frac{\pi}{2} = \frac{9\pi}{16}, \quad \frac{3\pi}{8}.$

$x = \frac{\pi}{16}, \quad \frac{9\pi}{16}, \quad \frac{3\pi}{8}.$

QUESTION 2

a. (i) $x = \mathrm{W}(6)$.

(ii) $x = \mathrm{W}(3)$ is the solution to the equation $xe^{3x} = 3 \Leftrightarrow xe^{3x} - 3 = 0$.

Using the graphics calculator to solve this equation: $x = 1 \cdot 0499$,

correct to four decimal places.

(iii) $xe^{3x} = 2$

Multiply both sides by 3:

$$\therefore (3x)e^{(3x)} = 6$$

$$\therefore 3x = \mathrm{W}(6)$$

$$\therefore x = \frac{1}{3}\mathrm{W}(6).$$

b. (i) $x + e^{x} = 2$

$$\therefore e^{x} = 2 - x$$

$$\therefore 1 = e^{-x}(2 - x).$$

(ii) $1 = e^{-x}(2 - x)$

Substitute $2 - x = t \Rightarrow x = 2 - t$:

$$\therefore 1 = e^{-(2-t)}t$$

$$= e^{-2}e^{t}t$$

$$\therefore e^{2} = e^{t}t.$$

(iii) The solution to $e^{2} = e^{t}t$ is $t = \mathrm{W}(e^{2})$.

Therefore: $2 - x = \mathrm{W}(e^{2})$

$$\therefore x = 2 - \mathrm{W}(e^{2}).$$

c. (i) f is required to be 1-to-1.

The largest value of a will therefore be equal to the $x-$coordinate of the turning point.

Option 1: Use the graphics calculator.

Option 2: Use calculus.

Let $f(x) = uv$ where $u = x$ and $v = e^{-x}$.

Then $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = -e^{-x}$.

Applying the **Product Rule:** $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

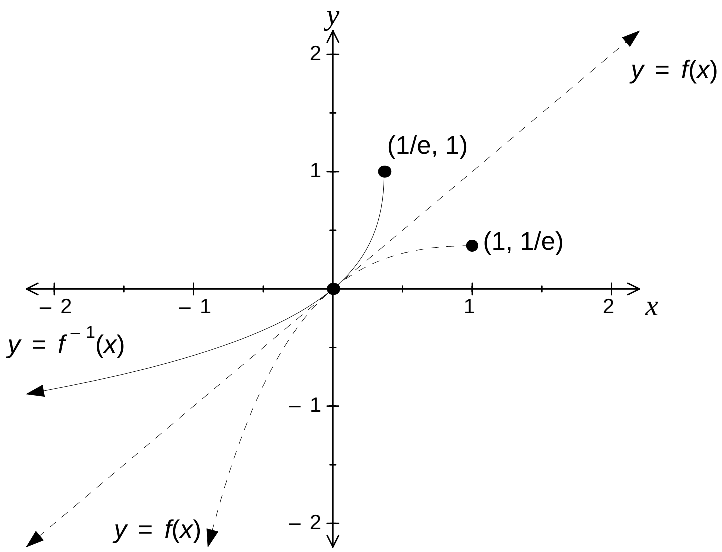
$\therefore \dfrac{dy}{dx} = (x)(-e^{-x}) + (e^{-x})(1) = e^{-x} - xe^{-x} = (1-x)e^{-x}$.

To find $x-$coordinate of stationary points solve $\dfrac{dy}{dx} = 0$: $(1-x)e^{-x} = 0$.

Use the null factor theorem: $(1-x) = 0 \therefore x = 1 \therefore a = 1$.

Note that $e^{-x} = 0$ has no real solution.

(ii)



(iii) Let $y = f^{-1}(x)$. Then:

$$x = ye^{-y}$$

$$\therefore -x = (-y)e^{(-y)}$$

$$\therefore -y = \mathrm{W}(-x)$$

$$\therefore y = -\mathrm{W}(-x).$$

$$f^{-1}(x) = -\mathrm{W}(-x).$$

(iv) Substitute $x = \frac{1}{e}$ into $f^{-1}(x) = -\mathrm{W}(-x)$: $f^{-1}\left(\frac{1}{e}\right) = -\mathrm{W}\left(-\frac{1}{e}\right)$.

From (ii): $f^{-1}\left(\frac{1}{e}\right) = 1$.

Therefore: $-\mathrm{W}\left(-\frac{1}{e}\right) = 1 \Rightarrow \mathrm{W}\left(-\frac{1}{e}\right) = -1$.

QUESTION 3

a. (i) Let $y = uv$ where $u = x^2 + \alpha x + \beta$ and $v = \sqrt{2x-3} = (2x-3)^{1/2}$.

Then $\dfrac{du}{dx} = 2x + \alpha$ and $\dfrac{dv}{dx} = (2x-3)^{-1/2} = \dfrac{1}{\sqrt{2x-3}}$.

Applying the **Product Rule:** $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$$\therefore \frac{dy}{dx} = (x^2 + \alpha x + \beta)\frac{1}{\sqrt{2x-3}} + \sqrt{2x-3}(2x+\alpha)$$

$$= \frac{x^2 + \alpha x + \beta}{\sqrt{2x-3}} + (2x+\alpha)\sqrt{2x-3}\,.$$

(ii) From part (i):

$$\frac{dy}{dx} = \frac{x^2 + \alpha x + \beta}{\sqrt{2x-3}} + (2x+\alpha)\sqrt{2x-3}$$

Write over a common denominator:

$$= \frac{x^2 + \alpha x + \beta}{\sqrt{2x-3}} + \frac{(2x+\alpha)\sqrt{2x-3}\sqrt{2x-3}}{\sqrt{2x-3}}$$

$$= \frac{x^2 + \alpha x + \beta + (2x+\alpha)(2x-3)}{\sqrt{2x-3}}$$

$$= \frac{x^2 + \alpha x + \beta + 4x^2 - 6x + 2\alpha x - 3\alpha}{\sqrt{2x-3}}$$

$$= \frac{5x^2 + x(3\alpha - 6) + (\beta - 3\alpha)}{\sqrt{2x-3}} \quad \text{where } q(x) = 5x^2 + x(3\alpha - 6) + (\beta - 3\alpha)\,.$$

b. Link to part (a): $\dfrac{2x(5x-9)}{\sqrt{2x-3}} = 2\dfrac{(5x^2-9x)}{\sqrt{2x-3}}$.

Let $q(x) = 5x^2 - 9x$: $5x^2 - 9x \equiv 5x^2 + x(3\alpha - 6) + (\beta - 3\alpha)$.

Therefore: $3\alpha - 6 = -9 \Rightarrow \alpha = -1$. (1)
$\beta - 3\alpha = 0$. (2)

Solve equations (1) and (2) simultaneously: $\beta - 3(-1) = 0 \Rightarrow \beta = -3$.

It follows that the derivative of $(x^2 - x - 3)\sqrt{2x-3}$ is $\dfrac{5x^2-9x}{\sqrt{2x-3}}$:

$$\frac{5x^2-9x}{\sqrt{2x-3}} = \frac{d}{dx}\left\{(x^2-x-3)\sqrt{2x-3}\right\}$$

$$\therefore 2\frac{(5x^2-9x)}{\sqrt{2x-3}} = 2\frac{d}{dx}\left\{(x^2-x-3)\sqrt{2x-3}\right\}$$

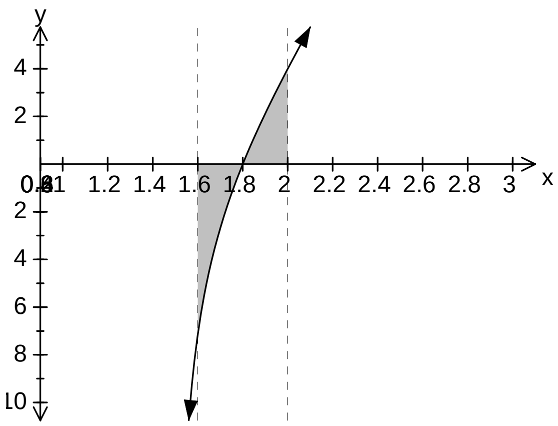
Anti-differentiate both sides with respect to x:

$$\therefore 2\int \frac{(5x^2-9x)}{\sqrt{2x-3}}\,dx = 2(x^2-x-3)\sqrt{2x-3}$$

$$\therefore \int \frac{2x(5x-9)}{\sqrt{2x-3}}\,dx = 2(x^2-x-3)\sqrt{2x-3}$$

where the arbitrary constant of anti-differentiation is omitted since only *an* anti-derivative is required.

c. (i)



The curve intersects the *x*-axis when $5x-9=0 \Rightarrow x=\dfrac{9}{5}$.

Therefore: Area $= -\displaystyle\int_{8/5}^{9/5} \frac{2x(5x-9)}{\sqrt{2x-3}}\,dx + \int_{9/5}^{2} \frac{2x(5x-9)}{\sqrt{2x-3}}\,dx$

$$= -2\left[(x^2-x-3)\sqrt{2x-3}\right]_{8/5}^{9/5} + 2\left[(x^2-x-3)\sqrt{2x-3}\right]_{9/5}^{2}\ .$$

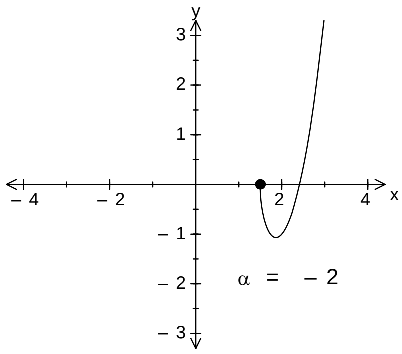
(ii) Using the numerical integration feature of the graphics calculator:

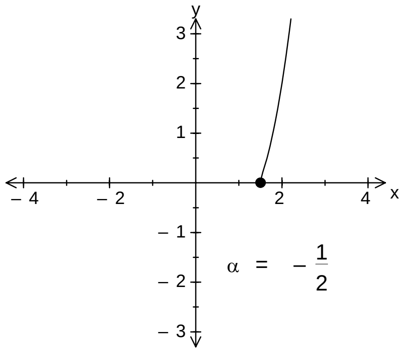
$$-\int_{8/5}^{9/5} \frac{2x(5x-9)}{\sqrt{2x-3}}\,dx + \int_{9/5}^{2} \frac{2x(5x-9)}{\sqrt{2x-3}}\,dx \approx -(-0\cdot 59211)+(0\cdot 41674)$$

$= 1\cdot 009$, correct to three decimal places.

Note: In order to avoid accumulation of rounding error, a greater degree of accuracy than required in the final answer is used during the calculation.

d. From the graphics calculator it is easily seen that, depending on the value of α, $f(x)=(x^2+\alpha x-1)\sqrt{2x-3}$ either has a minimum turning point or is always an increasing function and so has no turning point. Note that $\text{dom } f=\left[\frac{3}{2},\infty\right)$.





It is therefore required to find all exact values of α for which $y=f(x)$ has no turning point.

Link to part (a): $\beta=-1$

$$\therefore\ f'(x)=\frac{5x^2+x(3\alpha-6)+(-1-3\alpha)}{\sqrt{2x-3}}=\frac{5x^2+x(3\alpha-6)-(1+3\alpha)}{\sqrt{2x-3}}.$$

Solve $f'(x)=0$ to find the *x*-coordinate of the turning point:

$$0=\frac{5x^2+x(3\alpha-6)-(1+3\alpha)}{\sqrt{2x-3}}$$

$$\therefore 0=5x^2+x(3\alpha-6)-(1+3\alpha)$$

Use the quadratic formula:

$$\therefore x=\frac{6-3\alpha\pm\sqrt{(3\alpha-6)^2+20(1+3\alpha)}}{10}=\frac{6-3\alpha\pm\sqrt{9\alpha^2+24\alpha+56}}{10}.$$

Therefore $y=f(x)$ has no turning point when either:

$$9\alpha^2+24\alpha+56<0 \qquad (1)$$

or

$$\frac{6-3\alpha\pm\sqrt{9\alpha^2+24\alpha+56}}{10}<\frac{3}{2}, \qquad (2)$$

where inequation (2) follows from $\text{dom } f=\left[\frac{3}{2},\infty\right)$.

Equation (1) has no real solution.

From inequation (2): $\pm\sqrt{(9\alpha^2+24\alpha+56}<9+3\alpha$.

To solve this inequality, consider the solution to $\pm\sqrt{(9\alpha^2+24\alpha+56}=9+3\alpha$:

$$9\alpha^2+24\alpha+56=(9+3\alpha)^2$$

$$\therefore 9\alpha^2+24\alpha+56=81+54\alpha+9\alpha^2$$

$$\therefore \alpha=-\frac{5}{6}.$$

The solution to $-3\alpha\pm\sqrt{9\alpha^2+24\alpha+56}<9$ is therefore $\alpha>-\frac{5}{6}$.

Therefore $f(x)=(x^2+\alpha x-1)\sqrt{2x-3}$ is always an increasing function for $\alpha>-\frac{5}{6}$.

Solution 2

By definition the exact values of α such that $f'(x)>0$ for all $x\in\text{dom } f=\left[\frac{3}{2},\infty\right)$ are required.

Link to part (a): $\beta=-1$

$$\therefore f'(x)=\frac{5x^2+x(3\alpha-6)+(-1-3\alpha)}{\sqrt{2x-3}}=\frac{5x^2+x(3\alpha-6)-(1+3\alpha)}{\sqrt{2x-3}}.$$

It follows that exact values of α such that $5x^2+x(3\alpha-6)-(1+3\alpha)>0$ over the domain $\left[\frac{3}{2},\infty\right)$ are required. It is therefore required that the minimum value of the parabola $y=5x^2+x(3\alpha-6)-(1+3\alpha)$ over the domain $\left[\frac{3}{2},\infty\right)$ is always greater than zero. Depending on the value of α, the minimum value of this parabola will occur at either its minimum turning point or at its endpoint.

x-coordinate of minimum turning point:

$$\frac{dy}{dx}=0\Rightarrow 10x+(3\alpha-6)=0\ \therefore x=\frac{6-3\alpha}{10}.$$

The minimum value of $y=5x^2+x(3\alpha-6)-(1+3\alpha)$ will therefore occur at its minimum turning point when $\frac{6-3\alpha}{10}\geq\frac{3}{2}\Rightarrow\alpha\leq-3$ and its endpoint when $\alpha>-3$.

y-coordinate of minimum turning point:

$$y = 5\left(\frac{6-3\alpha}{10}\right)^2 + \left(\frac{6-3\alpha}{10}\right)(3\alpha - 6) - (1+3\alpha)$$

$$= \frac{(180 - 180\alpha + 45\alpha^2)}{100} - \frac{(36 - 36\alpha + 9\alpha^2)}{10} - (1+3\alpha)$$

$$= \frac{(180 - 180\alpha + 45\alpha^2) - (360 - 360\alpha + 90\alpha^2) - 100(1+3\alpha)}{100}$$

$$= \frac{-280 - 120\alpha - 45\alpha^2}{100} = -\frac{(56 + 24\alpha + 9\alpha^2)}{20} < 0 \text{ for } \alpha \le -3.$$

Therefore $5x^2 + x(3\alpha - 6) - (1+3\alpha)$ is never always greater than zero over the domain $\left[\frac{3}{2}, \infty\right)$ for $\alpha \le -3$.

y -coordinate of endpoint:

$$y = 5\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)(3\alpha - 6) - (1+3\alpha) = \frac{6\alpha+5}{4}\begin{cases} > 0 & \text{for} \quad \alpha > -\frac{5}{6} \\ \le 0 & \text{for} \quad \alpha \le -\frac{5}{6} \end{cases}$$

Therefore $5x^2 + x(3\alpha - 6) - (1+3\alpha)$ is always greater than zero over the domain $\left[\frac{3}{2}, \infty\right)$ for $\alpha > -\frac{5}{6}$.

Therefore $f(x) = (x^2 + \alpha x - 1)\sqrt{2x-3}$ is always an increasing function for $\alpha > -\frac{5}{6}$.

QUESTION 4

a. (i) Probability density function: $f(x) = \frac{1}{5} x e^{-x^2/10}$.

$$E(X) = \int_0^{+\infty} \frac{1}{5} x^2 e^{-x^2/10}\, dx \approx \int_0^{100} \frac{1}{5} x^2 e^{-x^2/10}\, dx \approx 2 \cdot 802496 .$$

Note: The use of an upper integral limit of 100 is a valid approximation for obtaining an answer correct to the required accuracy.

The expected value of the length of life of an aircraft guidance system is equal to 2802 hours, correct to the nearest hour.

(ii) $$Var(X) = \int_0^{+\infty} \frac{1}{5} x e^{-x^2/10} (x - 2 \cdot 802496)^2\, dx$$

$$\approx \int_0^{100} \frac{1}{5} x e^{-x^2/10} (x - 2 \cdot 802496)^2\, dx \approx 2 \cdot 146018$$

$$\therefore sd(X) \approx \sqrt{2 \cdot 146018} \approx 1 \cdot 46493 .$$

Note: The use of an upper integral limit of 100 is a valid approximation for obtaining an answer correct to the required accuracy.

The standard deviation of the length of life of an aircraft guidance system is equal to 1465 hours, correct to the nearest hour.

b. $\Pr(X > 4) = \int_4^{+\infty} \frac{1}{5} x e^{-x^2/10}\, dx \approx \int_4^{100} \frac{1}{5} x e^{-x^2/10}\, dx = 0 \cdot 2019$ correct to four decimal places.

Note: The use of an upper integral limit of 100 is a valid approximation for obtaining an answer correct to the required accuracy.

c. Let Y denote the random variable *number of guidance systems that fail*.

Then Y ~ Binomial($p = 1 - 0 \cdot 2018965 = 0 \cdot 7981035$, $n = 3$).

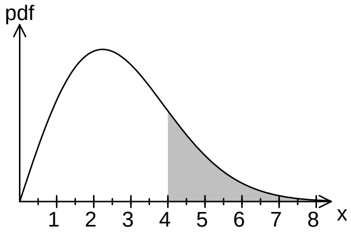
Note: In order to avoid accumulation of rounding error, a greater degree of accuracy than required in the final answer is used during the calculation.

$\Pr(Y = 1) = {}^3C_1 (0 \cdot 7981035)^1 (1 - 0 \cdot 7981035)^2 = 0 \cdot 0976$, correct to four decimal places.

Alternatively, from the graphics calculator:

$\Pr(Y = 1) =$ binompdf(3, $0 \cdot 7981035$, 1) $= 0 \cdot 0976$.

d. $\Pr(X < 5 \mid X > 4) = \dfrac{\Pr(4 < X < 5)}{\Pr(X > 4)}$.



From part (b): $\Pr(X > 4) \approx 0 \cdot 2018965$.

$$\Pr(4 < X < 5) = \int_4^5 \frac{1}{5} x e^{-x^2/10}\, dx = 0 \cdot 119811.$$

Therefore: $\Pr(X < 5 \mid X > 4) \approx \dfrac{0 \cdot 119811}{0 \cdot 2018965} = 0 \cdot 5934$, correct to four decimal places.

e. Find correct to the nearest hour the expected length of life of an aircraft guidance system that has already lasted for four thousand hours.

A conditional expected value is required:

$$E(X \mid X > 4) = \int_4^{+\infty} x f(x \mid X > 4)\, dx.$$

Conditional probability density function:

$$f(x \mid X > 4) = \begin{cases} \dfrac{\frac{1}{5} x e^{-x^2/10}}{\Pr(X > 4)} \approx \dfrac{\frac{1}{5} x e^{-x^2/10}}{0 \cdot 2018965} = \dfrac{x e^{-x^2/10}}{1 \cdot 0094825}, & X > 4 \\ 0 & \text{otherwise} \end{cases}$$

Therefore:

$$E(X \mid X > 4) = \int_4^{+\infty} \frac{x^2 e^{-x^2/10}}{1 \cdot 0094825}\, dx \approx \frac{5 \cdot 069785}{1 \cdot 0094825} = 5 \cdot 022162.$$

Note: In order to avoid accumulation of rounding error, a greater degree of accuracy than required in the final answer is used during the calculation.

The expected length of life of an aircraft guidance system that has already lasted for four thousand hours is equal to 5022 hours, correct to the nearest hour.

BONUS QUESTION

QUESTION 1

a. $y = f(g(2)) = f(-1) = 10$.

b. To find $\dfrac{dy}{dx}$ use the Chain Rule:

Let $w = g(x)$ so that $y = f(w)$.

Then $\dfrac{dy}{dw} = f'(w)$ and $\dfrac{dw}{dx} = g'(x)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{dw} \times \dfrac{dw}{dx} = f'(w)g'(x) = f'(g(x))g'(x)$.

Substitute $x = 2$ into $\dfrac{dy}{dx}$:

$\dfrac{dy}{dx} = f'(g(2))g'(2) = f'(-1)g'(2) = (-1)(-4) = 4$.

c. (i) It is required that $\text{ran } h \subseteq \text{dom } f$.

$\text{dom } f: \ 3 - x \geq 0 \Rightarrow x \leq 3$.

It is therefore required that $h(x) \leq 3$:

$h(x) \leq 3$

$\therefore 1 - \log_e(2x-1) \leq 3$

$\therefore \log_e(2x-1) \geq -2$

$\therefore 2x - 1 \geq e^{-2}$

$\therefore x \geq \dfrac{e^{-2}+1}{2}$.

(ii) $y = 2 + 4\sqrt{3 - [1 - \log_e(2x-1)]}$

$y = 2 + 4\sqrt{2 + \log_e(2x-1)}$.

(iii) $2+4\sqrt{2+\log_e(2\beta-1)}=10$

$\therefore \sqrt{2+\log_e(2\beta-1)}=2$

$\therefore 2+\log_e(2\beta-1)=4$

$\therefore \log_e(2\beta-1)=2$

$\therefore 2\beta-1=e^2$

$\therefore \beta=\dfrac{e^2+1}{2}$.