INSIGHT

Trial Exam Paper

2006

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

Question 1

Which one of the following functions does **not** have an inverse function?

A. $f:[-3,1)\to R,\ f(x)=\sqrt{x+3}$

B. $g:R^{+}\to R,\ g(x)=\dfrac{1}{x^{3}}+3$

C. $h:R\to R,\ h(x)=x^{5}$

D. $k:[0,\infty)\to R,\ k(x)=x^{2}+1$

E. $\boldsymbol{m:R^{+}\to R, m(x)=|2x-5|}$

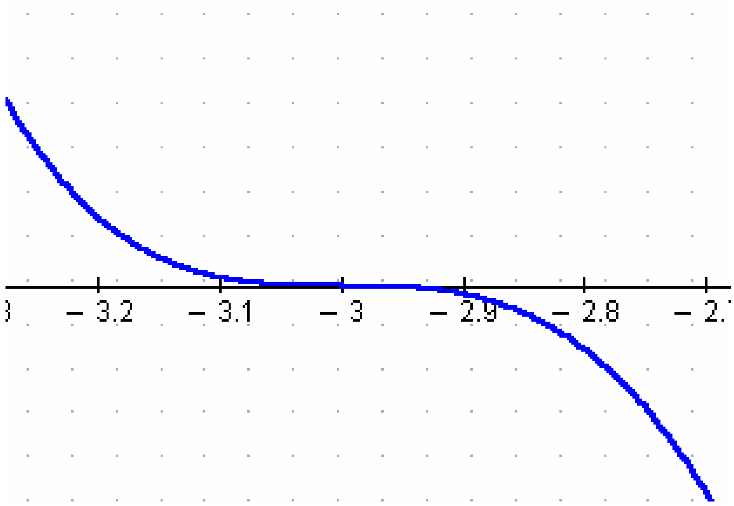
Answer is E

Worked solution

- For an inverse function to exist, the function must be one-to-one, and E is not one-to-one.

Question 2

A polynomial function p has degree 4. A part of its graph, near the point on the graph with the coordinates $(-3, 0)$, is shown below.



Which one of the following could be the rule for the fourth degree polynomial p?

A. $p(x)=(x+3)^{4}$

B. $p(x)=x^{3}(x+3)$

C. $\boldsymbol{p(x)=(x-1)(x+3)^{3}}$

D. $p(x)=-x(x+3)^{3}$

E. $p(x)=x^{2}(x+3)^{2}$

Answer is C

Worked solution

- The graph has a point of inflection at $x = -3$ and it would be possible to have an x-intercept at $x = 1$ – this can be checked by sketching the graph.

Question 3

The maximal domain, D, of the function $f: D \rightarrow R$ with the rule $f(x) = \log_2((x-2)^2)$ is

A. $R \setminus \{0\}$

B. $\boldsymbol{R \setminus \{2\}}$

C. R

D. $(2, \infty)$

E. $(-\infty, 2)$

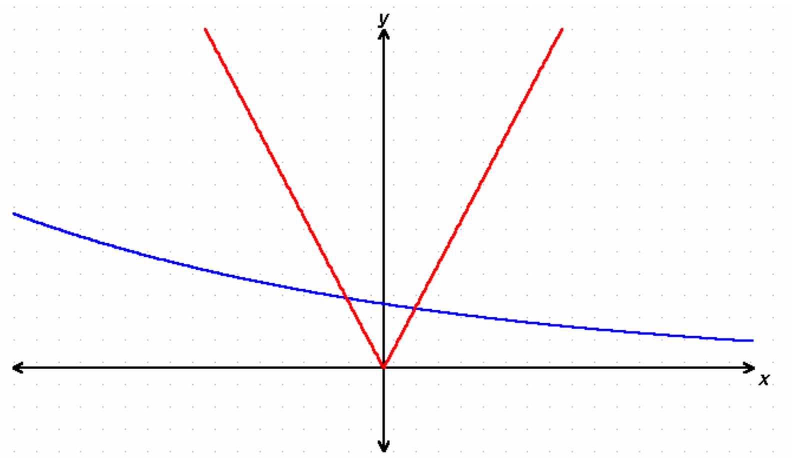
Answer is B

Worked solution

- $(x-2)^2$ needs to be $\neq 0$ so $x \neq 2$.

Question 4

Part of the graphs of the functions with equations $y = |3x|$ and $y = e^{-0.4x}$ are shown below.



The solution of the equation $|3x| = e^{-0.4x}$ for $x < 0$ is closest to

A. 0.30

B. 0.29

C. 1.17

D. -0.38

E. $\boldsymbol{-0.39}$

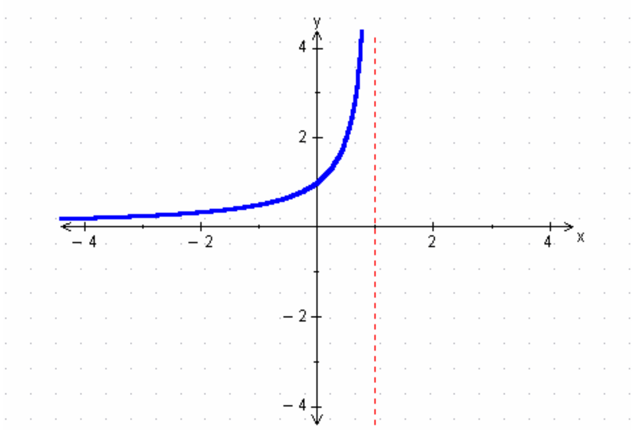
Answer is E

Worked solution

- A graphics calculator can be used to calculate the intersection of $y = -3x$ with $y = e^{-0.4x}$ as being -0.39.

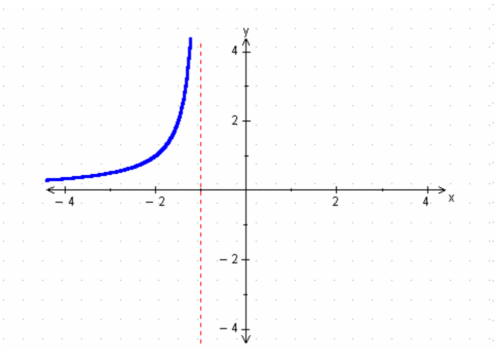
Question 5

Part of the graph with the function with rule $y = f(x)$ is shown below.

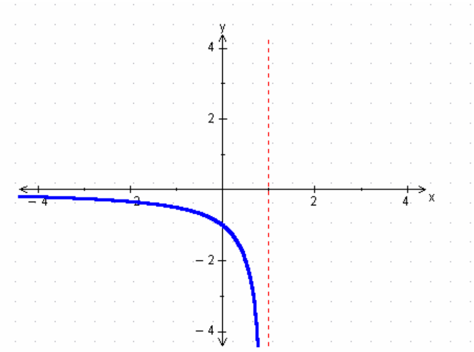


Which one of the following is most likely to be the corresponding part of the graph of the function with rule $y = f(-x)$

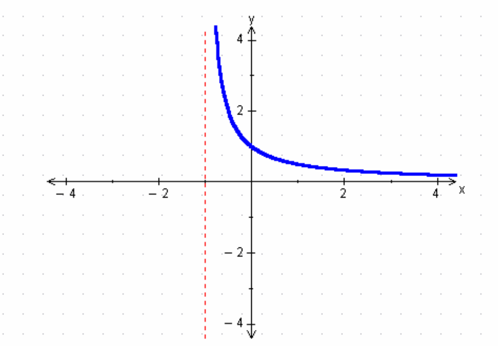
A.



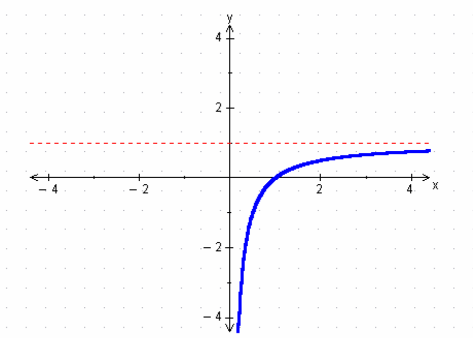
B.



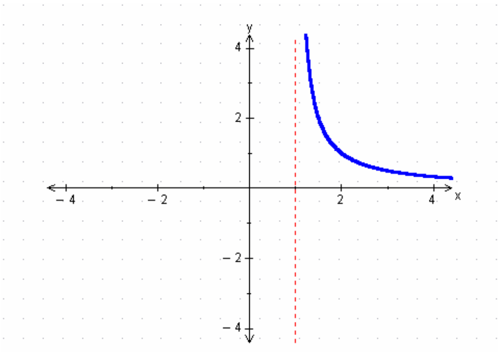
C.



D.



E.



Answer is C

Worked solution

- $f(-x)$ is a reflection in the y-axis.

Question 6

The **linear** factors of $x^4 + x^2 - 2$ over *R* are

A. $x-1$

B. $\boldsymbol{x-1, x+1}$

C. $x-1,\ x+1,\ x^2+2$

D. $x-1,\ x+1,\ x-\sqrt{2},\ x+\sqrt{2}$

E. $x-1,\ x+1,\ x,\ x+2$

Answer is B

Worked solution

- $x^4 + x^2 - 2$ factorises to give $(x+1)(x-1)(x^2+2)$, and $(x+1)$ and $(x-1)$ are the only linear factors.

Question 7

The number of solutions to the equation $(x^2+a)(x^3-b)(x+c)=0$ where *a, b, c* $\in R^+$ is

A. 6

B. 5

C. 4

D. 3

E. **2**

Answer is E

Worked solution

- $(x^3-b)=0$ will give one solution and $(x+c)=0$ will give the other.

Question 8

$\log_7 9$ is equal to

A. $\dfrac{\log_e 7}{\log_e 9}$

B. $\boldsymbol{\dfrac{\log_e 9}{\log_e 7}}$

C. $\log_e(\frac{9}{7})$

D. $\log_e 2$

E. $\log_e 9 - \log_e 7$

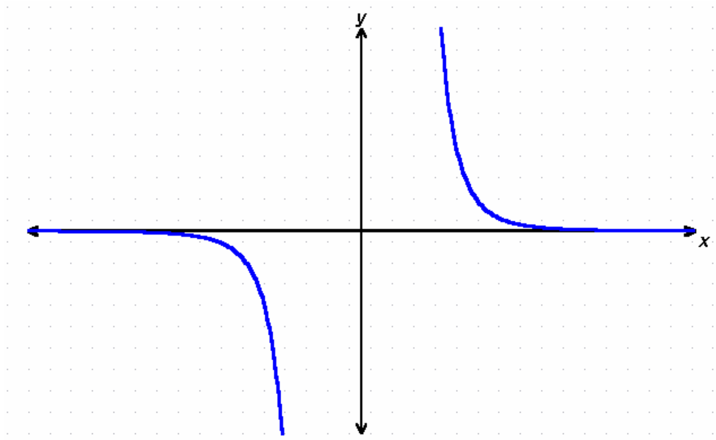
Answer is B

Worked solution

- Use the change of base rule.

SECTION 1 – continued

Question 9



The rule for the graph shown could be

A. $y = x^{\frac{1}{5}}$

B. $y = x^{\frac{5}{2}}$

C. $y = x^{\frac{2}{5}}$

D. $y = \dfrac{1}{x^5}$

E. $y = \left| x^{\frac{1}{5}} \right|$

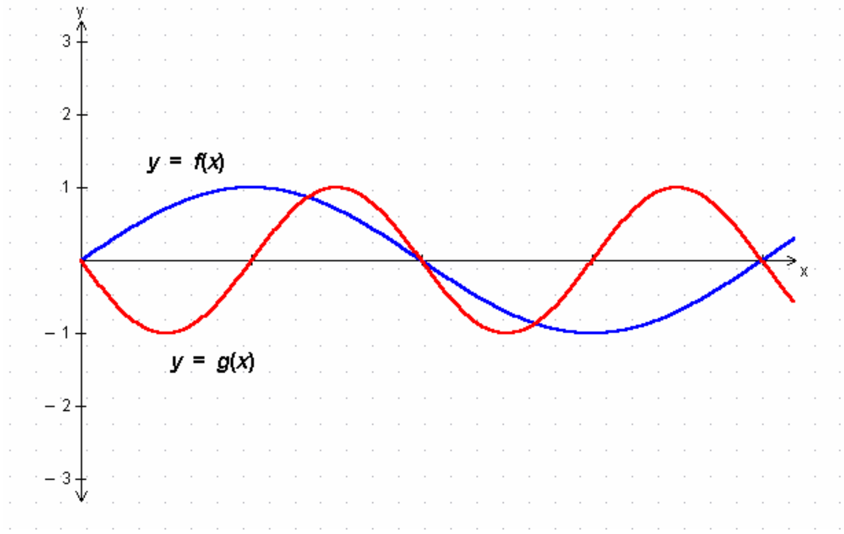
Answer is D

Worked solution

- Either know the properties of each type of graph, or use a graphics calculator to sketch all the options.

Question 10

The diagram below shows the graphs of two circular functions, *f* and *g*.



The graph of the function with the equation $y = f(x)$ is transformed into the graph of the function with equation $y = g(x)$ by

A. a reflection in the x-axis and a dilation of factor 2 from the y-axis.

B. a reflection in the y-axis and a dilation of factor 2 from the x-axis.

C. a reflection in the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

D. a reflection in the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

E. a reflection in the y-axis and a dilation of factor π from the y-axis.

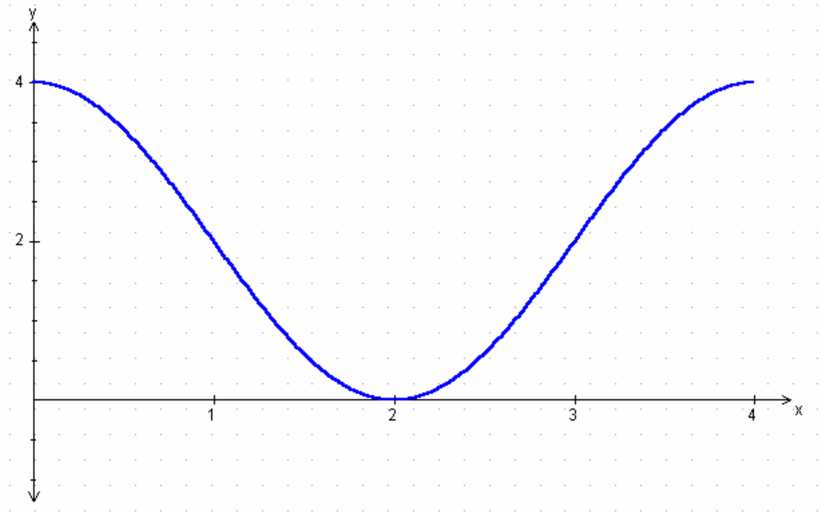
Answer is D

Worked solution

- The graph has been reflected in the x-axis and the period has been halved.

Question 11

The diagram below shows one cycle of the graph of a circular function.



A possible equation for the function whose graph is shown is

A. $y = 4\cos(4x) + 2$

B. $y = 2\cos(2x) + 2$

C. $y = 2\cos(\frac{1}{2}x) + 2$

D. $\boldsymbol{y = 2\cos(\frac{\pi}{2}x) + 2}$

E. $y = 4\cos(8\pi x) + 2$

Answer is D

Worked solution

- The amplitude is 2, the graph has been shifted up 2 units, and the period is 4 so $n = \frac{\pi}{2}$.

Question 12

The depth of water near the Sorrento pier changes with the tides and can be given by the rule

$$d(t) = 8 - 3\sin(\frac{4\pi t}{25} + \frac{3\pi}{2})$$

where t is the time in hours after high tide and d is the depth in metres. A high tide occurred at 3 a.m. The time of the next **low tide** is

A. **9:15 a.m.**

B. 3:50 a.m.

C. 9:25 a.m.

D. 3:30 a.m.

E. 3:50 p.m.

Answer is A

Worked solution

- The period is 12.5 hours, so half a period is 6.25 hours. 6.25 hours on from 3 a.m. is 9:15 a.m.

Question 13

If $y = |\cos(2x)|$, then the rate of change of y with respect to x at $x = k$, $\frac{\pi}{2} < x < \frac{3\pi}{4}$, is

A. $-2\sin(2k)$

B. $\mathbf{2\sin(2k)}$

C. $-\sin(2k)$

D. $-\cos(2k)$

E. $\frac{1}{2}\sin(2k)$

Answer is B

Worked solution

- In the domain $\frac{\pi}{2} < x < \frac{3\pi}{4}$, $y = -\cos(2x)$, so $\frac{dy}{dx} = 2\sin(2x)$.

Question 14

An ice-block melts and forms a circular puddle on the floor. The radius of the puddle increases at a rate of 3 cm/min. The rate at which the area is increasing, in cm^2/min, when the radius is 2 cm is

A. 6π

B. $\mathbf{12\pi}$

C. 3π

D. $\frac{12}{\pi}$

E. 24π

Answer is B

Worked solution

$A = \pi r^2$

$\Rightarrow \frac{dA}{dr} = 2\pi r$

$\Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 3 \times 2\pi r = 6\pi r$

At $r = 2, \frac{dA}{dt} = 12\pi$

Question 15

If $y = \log_e(\sin(3x))$, then $\frac{dy}{dx}$ is equal to

A. $3\tan(3x)$

B. $-3\tan(3x)$

C. $\frac{1}{\tan(3x)}$

D. $\boldsymbol{\frac{3}{\tan(3x)}}$

E. $3\cos(3x)$

Answer is D

Worked solution

$$\frac{dy}{dx} = \frac{1}{\sin(3x)} 3\cos(3x)$$
$$= \frac{3}{\tan(3x)}$$

Question 16

If u is a function of x and if $y = u(x).x^{\frac{2}{3}}$, then the rate of change of y with respect to x when $x = 8$ is equal to

A. $\frac{2}{3}u(8)x^{\frac{-1}{3}}$

B. $\frac{1}{3}u'(8)$

C. $\boldsymbol{4u'(8) + \frac{1}{3}u(8)}$

D. $4u'(8) + 4u(8)$

E. $4u(8)$

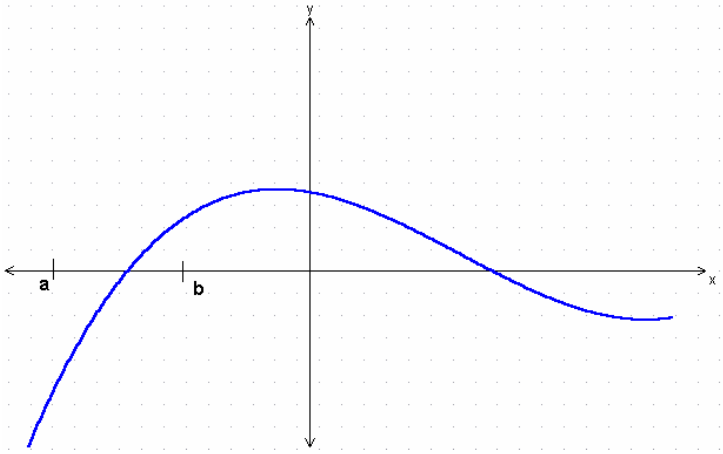
Answer is C

Worked solution

- Differentiating using the chain rule gives $\frac{dy}{dx} = u'(x)x^{\frac{2}{3}} + \frac{2}{3}u(x)x^{\frac{-1}{3}}$, then we need to evaluate at $x = 8$.

Question 17

Part of the graph of a function *f* is shown below.



Let g be a function such that $g'(x) = f(x)$.

Over the interval *(a,b),* the graph of g will have a

A. point of inflection.

B. positive gradient.

C. local maximum value.

D. **local minimum value.**

E. negative gradient.

Answer is D

Worked solution

- The graph shows that the gradient goes from negative to zero to positive, therefore a local minimum exists.

Question 18

If $f'(x) = 3\cos(2x - 3) + 1$, then $f(x)$ could be equal to

A. $-6\sin(2x - 3) + x$

B. $\mathbf{\frac{3}{2}\sin(2\boldsymbol{x} - 3) + \boldsymbol{x}}$

C. $\frac{2}{3}\sin(2x - 3) + x$

D. $\frac{-3}{2}\sin(x^2 - 3x) + x$

E. $-6\sin(2x)$

Answer is B

Worked solution

- Antidifferentiate and cos goes back to sin and then divide by the derivative of the brackets (from formula sheet, $\int \cos(ax) = \frac{1}{a}\sin(ax) + c$).

Question 19

If $\int_1^2 f(x)dx = 4$, then $\int_2^3 2f(x-1)dx$ is equal to

A. **8**

B. 7

C. 6

D. 4

E. 3

Answer is A

Worked solution

- The graph has moved to the right and the endpoints have been adjusted so there is no change to area. The graph has then been dilated by a factor of 2 from the *x*-axis, doubling the area.

Question 20

The number of defective calculators in a box of calculators ready for sale is a random variable with a binomial distribution with mean 20 and variance of 14.

If a calculator is drawn from the box, the probability that it is defective is

A. 0.1

B. 0.2

C. **0.3**

D. 0.5

E. 0.7

Answer is C

Worked solution

- Mean = np and variance = npq. This gives q = 0.7, so p = 0.3. Therefore the probability that a calculator is defective (success signifies 'defective' in this case) is 0.3.

Question 21

The random variable X has the following probability distribution, where $0 < p < \frac{1}{2}$.

x	0	1
$\Pr(X = x)$	$2p$	$1-2p$

The standard deviation of X is

A. $1-2p$

B. $p(1-2p)$

C. $2p(1-2p)$

D. $\boldsymbol{\sqrt{2p(1-2p)}}$

E. $\sqrt{p(1-p)}$

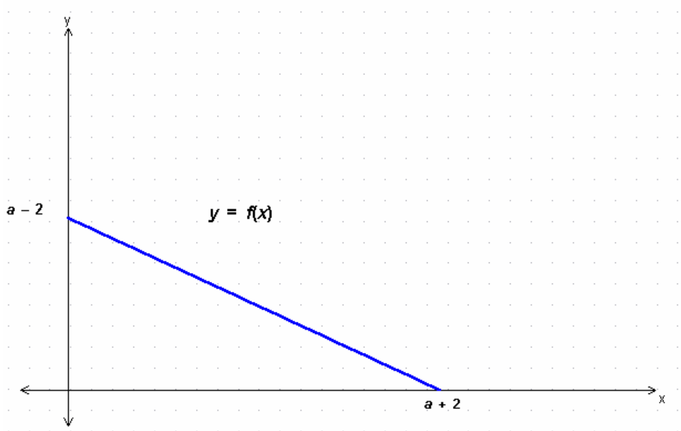
Answer is D

Worked solution

- There are only 2 outcomes, so the distribution is binomial. Therefore,
  $sd = \sqrt{npq} \ \ with \ \ n = 1, \ p = 1 - 2p, q = 2p$

Question 22

The graph shown below represents a probability density function.



The value of a is

A. $\frac{1}{4}$

B. 3

C. $\sqrt{5}$

D. $\sqrt{\mathbf{6}}$

E. 6

Answer is D

Worked solution

- Since this is a probability density function, the area under the triangle is 1. Therefore using
  $\text{area of triangle} = \frac{1}{2}bh$

  $\frac{1}{2}(a-2)(a+2) = 1$

  $a^2 - 4 = 2$

  $a^2 = 6$

END OF SECTION 1

SECTION 2

Question 1

The team from 'Frontyard Frenzy', a garden renovation television show, are doing a garden makeover. As part of the makeover they are constructing a wire wall sculpture. Two identical pieces of curved wire are to be used. Each piece has a shape that can be defined by the rule

$$y_w = \frac{1}{2000}x(x-10)(x^2-60x+910) \text{ for } x \in [0,38]$$

where all measurements are in centimetres.

1a. Show that $x^2 - 60x + 910 > 0$.

Worked solution

$\Delta = b^2 - 4ac = -40$ and the curve is upright, therefore $x^2 - 60x + 910 > 0$

Marks

- 1 mark for identifying Δ
- 1 mark for identifying that the curve is upright

2 marks

1b. Find the x-intercepts of the graph of $y_w = \dfrac{1}{2000}x(x-10)(x^2-60x+910)$.

Answer

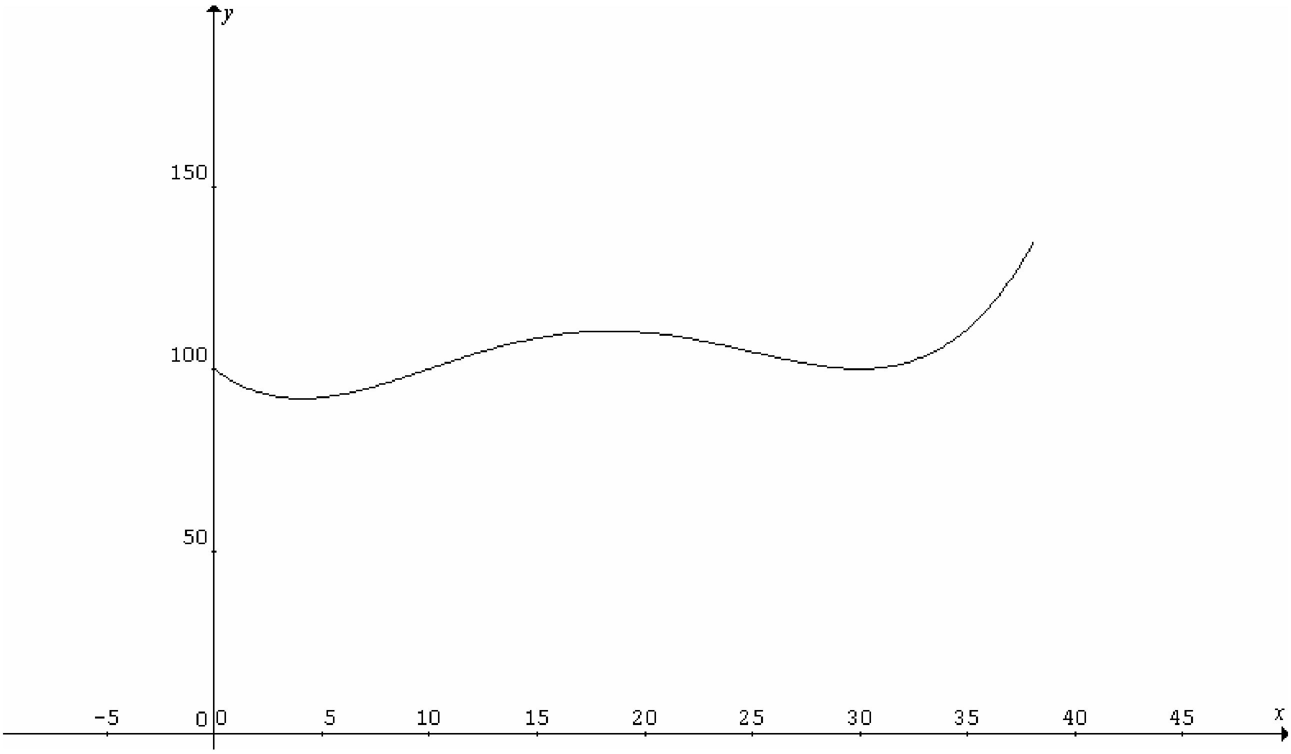
$x = 0$ or $x = 10$

Marks

- 1 mark for getting both answers correct

1 mark

One piece of wire (wire a, shown by the curve y_a) is to be attached to the wall with the starting point raised 100 cm off the ground as shown in the graph below.



1c. State the equation of the assembled piece of wire, y_a, as shown in the diagram above.

Answer

$$y_a = \frac{1}{2000}x(x-10)(x^2-60x+910)+100$$

Marks

- 1 mark for the correct equation

1 mark

The other piece of wire is attached to the wall and is positioned so that it is a reflection of wire a in the line $y = 120$. Its equation is given as $y_b = 140 - \frac{1}{2000}x(x-10)(x^2-60x+910)$.

1d. State the transformations (in the correct order) involved in producing y_b from y_w.

Answer

The transformations are: a reflection in the *x*-axis, followed by a translation of 140 units up

Marks

- 1 mark for stating the correct reflection
- 1 mark for stating translation up of 140 units
- If order is not correctly given, deduct one mark

2 marks

1e. The two wires are to be secured to the wall at the endpoints. State the coordinates of the endpoints for both pieces of wire (correct to two decimal places).

Answer

Endpoints are (0,140), (0,100), (38,100.63) and (38, 139.37)

Marks

- 2 marks for getting all four points correct and to two decimal places
- Deduct one mark for any incorrect point

2 marks

1f. The two wires intersect at one point in the domain. Find the point of intersection (correct to two decimal places).

Answer

Point of intersection is (35.78, 120)

Marks

- 1 mark for the correct coordinate – must be to two decimal places
- Stating (35.77, 120) gets zero marks

1 mark

It is decided to support the wires by placing vertical slats at selected intervals between the two curved wires.

1g. One vertical slat is to be placed at $x = 28$ cm . Find the minimum length of this slat (correct to two decimal places).

Answer

At $x = 28$, the y values for the wires are 136.472 and 103.528 – a difference of 32.94 cm. Therefore, the slat must be a minimum of 32.94 cm in length.

Marks

- 1 mark for getting both the y-values correct
- 1 mark for getting the difference correct

2 marks

1h. i. Two slats of length 10 cm are used. Where along the wires will slats of this length need to be positioned (correct to two decimal places)?

Worked solution

Using a graphics calculator, set up

$$y_3 = y_b - y_a$$

(the difference of the two y equations).

Then find the intersection of this new y_3 line with the lines $y = 10$ and $y = -10$. This gives intersection points of (36.47, -10) and (34.94, 10). So the slats should be placed at $x = 36.47$ cm and $x = 34.94$ cm.

Marks

- 1 mark for each of the 2 correct x values

1h. ii. Find the area of the region enclosed by the two curves and the two vertical slats of length 10 cm (correct to two decimal places).

Answer

It is necessary to calculate the area bounded by the curves from the first slat at $x = 34.94$ to the second at $x = 35.78$. The curves intersect at 35.78 and at this point change over from top to bottom (see below). We need to find the area from $x = 34.94$ to $x = 35.78$ and then from $x = 35.78$ to $x = 36.47$. The question is worth 2 marks, so just giving the final answer is not sufficient – evidence must be shown of area calculation and determination of the regions.

$$Area = \int_{34.94}^{35.78} y_b - y_a dx + \int_{35.78}^{36.47} y_a - y_b dx = 7.74 \text{ sq units}$$

Marks

- 1 mark for the correct integral or other correct working
- 1 mark for the correct answer, correct to 2 decimal places

2 + 2 = 4 marks

Total 15 marks

Question 2

Ethan, a curious young boy, is fascinated by his front-loading washing machine. He likes to watch it go through all the cycles and is particularly interested as to how fast it revolves when on spin cycle. He decides to place a red marker on the inside of the bowl and watches the red marker revolve around the edge of the bowl. The red marker is at its lowest point when the cycle begins, and the bowl is 70 cm in diameter. He times the spin cycle and notes that the red marker has gone around 5 times in 4 seconds.

The vertical position of the red marker can be modelled by the equation

$$y = A\cos nt + b$$

where y is the vertical position in centimetres from the starting point and t is the time in seconds. Assume the bowl spins continuously in one direction.

2a. State the period and show that $n = \frac{5\pi}{2}$.

Worked solution

- 5 revolutions in 4 seconds is equivalent to 1 revolution in 0.8 seconds, so the period is 0.8 seconds.

Answer

0.8 seconds

$$p = \frac{2\pi}{n} = 0.8$$

$$\Rightarrow n = \frac{2\pi}{0.8}$$

$$= 2.5\pi$$

$$= \frac{5\pi}{2}$$

Marks

- 1 mark for the correct period.
- 1 mark for working to show that $n = \frac{5\pi}{2}$

2 marks

SECTION 2 – continued

2b. Show that the value of A is −35 and b is 35

Worked solution

Set up two equations and solve simultaneously: we know that at $t = 0$, $y = 0$ and at half the period (halfway through the cycle) $t = 0.4$, $y = 70$. This gives –

$$t = 0, y = 0 \quad \Rightarrow b + A = 0$$
$$t = 0.4, y = 70 \Rightarrow b - A = 70$$

to give $A = -35$ and $b = 35$.

Other appropriate working is also acceptable.

Marks

- 1 mark for giving equations
- 1 mark for solving to get values of A and b

2 marks

2c. If the washing machine spins continuously for three minutes, explain why the marker will be at its lowest point at the start of every minute.

Worked solution

1 minute = 60 seconds, which is a whole number multiple of 0.8:

75×0.8 seconds

$\equiv 75$ full cycles

Therefore, the marker is at the lowest point at the start of each cycle.

Marks

- 1 mark for showing that the period divides exactly into one minute
- 1 mark for stating the cycle begins at lowest point

2 marks

A tumble dryer with diameter of 40 cm is mounted directly above the washing machine. Ethan continues his experiment by placing a blue marker on the internal drum of the dryer. The internal drum of the dryer moves so that it takes 1.2 seconds to complete one cycle. He starts both machines at the same time with both markers at their lowest points and the blue marker 120 cm above the red marker.

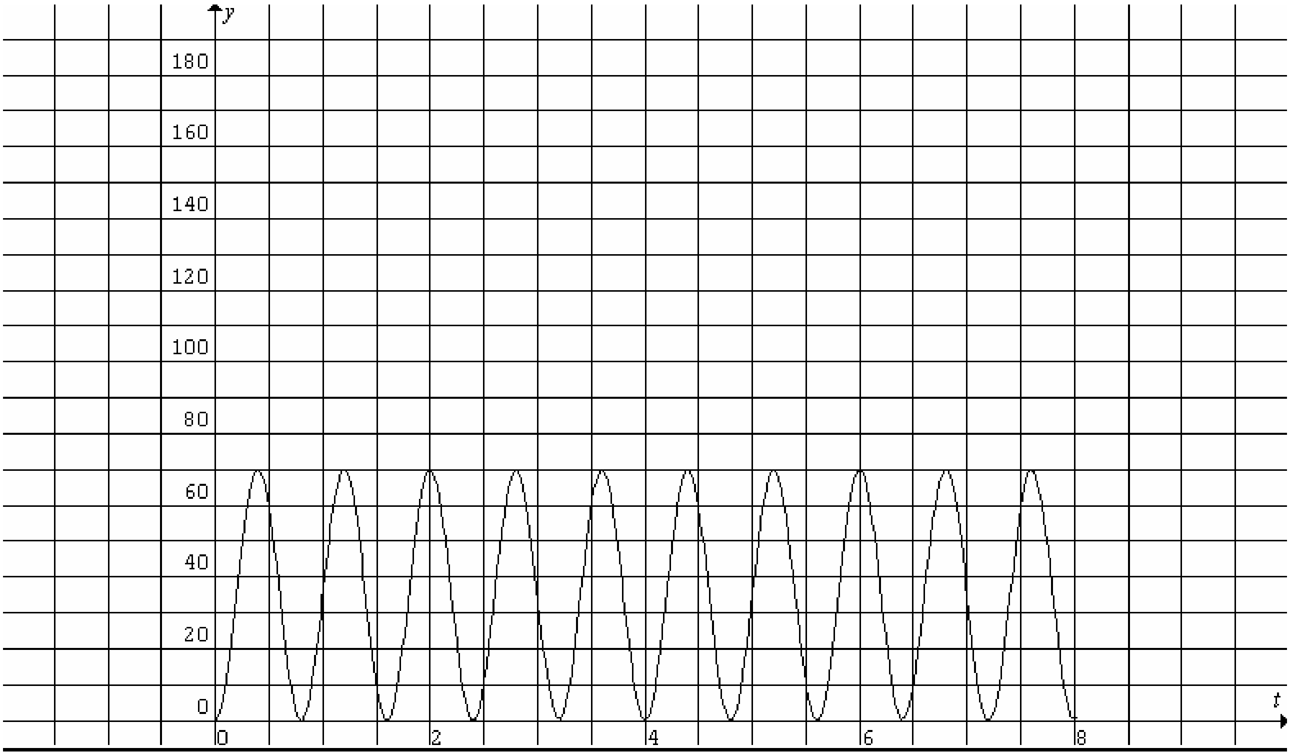
Both machines spin continuously in one direction. The equation for the vertical position of the blue marker, in centimetres, above the starting position of the red marker, is given as

$$y = 140 - 20\cos\left(\frac{5\pi t}{3}\right)$$

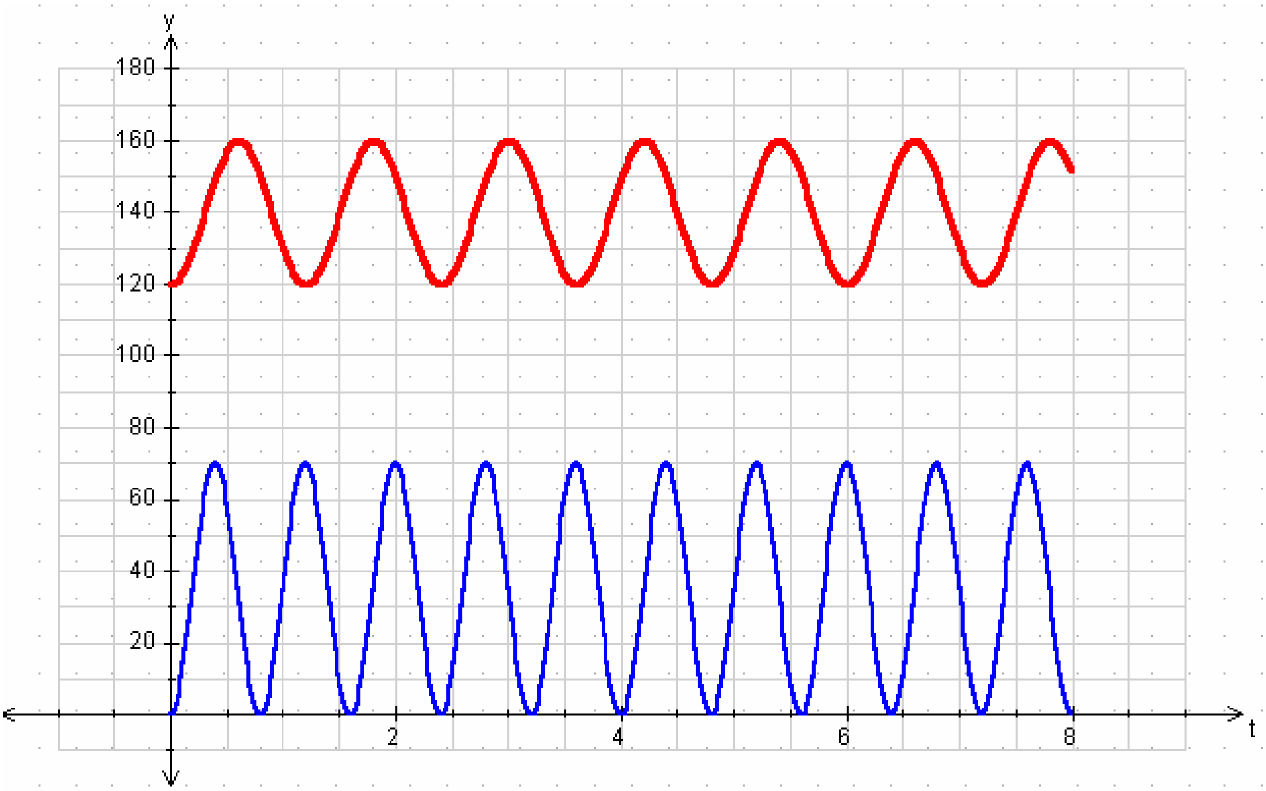
where t is the time in seconds since both machines started to spin.

The graph of the position of the red marker is given below for $t \in [0,8]$.

2d. On the axes given, sketch the graph of the position of the blue marker for $t \in [0,8]$



Answer



Marks

- 1 mark for getting the correct amplitude and centre line
- 1 additional mark for getting graph completely correct

2 marks

2e. i. Ethan notices that the markers come closest together three times in the first eight seconds. Find these three times.

Answer

From the graphs and the table menu (on the graphics calculator)

$t = 1.2, 3.6, 6$

Marks

- 2 marks for getting all three times correct
- If only two correct times are given then award 1 mark

The rule T_n gives the nth time when the two markers are closest together . T_n is defined as $T_n = a + bn$

2e. ii. Write an equation in terms of *a* and *b* for T_1 and T_2.

Answer

$T_1 = a + b \quad = 1.2$

$T_2 = a + 2b = 3.6$

Marks

- 1 mark for each correct equation

2e. iii. Solve the equations simultaneously to find *a* and *b* and hence state the tenth time when the markers are closest together.

Answer

$T_2 - T_1 \Rightarrow b = 2.4, \; a = -1.2$

gives $T_{10} = -1.2 + 2.4 \times 10 = 22.8$

Marks

- 1 mark for getting *a*, *b* correct
- 1 mark for getting *T* correct

2 + 2 + 2 = 6 marks

2f. i. Write down an expression, in terms of *t*, for the rate of change of the vertical position of the blue marker with respect to time.

Answer

From the equation for the position of the blue marker,

$$\frac{dy}{dt} = 20.\frac{5\pi}{3}\sin(\frac{5\pi t}{3})$$
$$= \frac{100\pi}{3}\sin(\frac{5\pi t}{3})$$

2f. ii. At what rate (in cm/s), correct to two decimal places, is the vertical position of the blue marker changing when $t = 1$?

Worked solution

$$\frac{dy}{dt} = \frac{100\pi}{3}\sin(\frac{5\pi t}{3})$$

$$\frac{dy}{dt} \text{ (at } t = 1) = -90.69$$

or from graphics calculator.

Marks

- 1 mark for the correct derivative
- 1 mark for the correct rate at $t = 1$

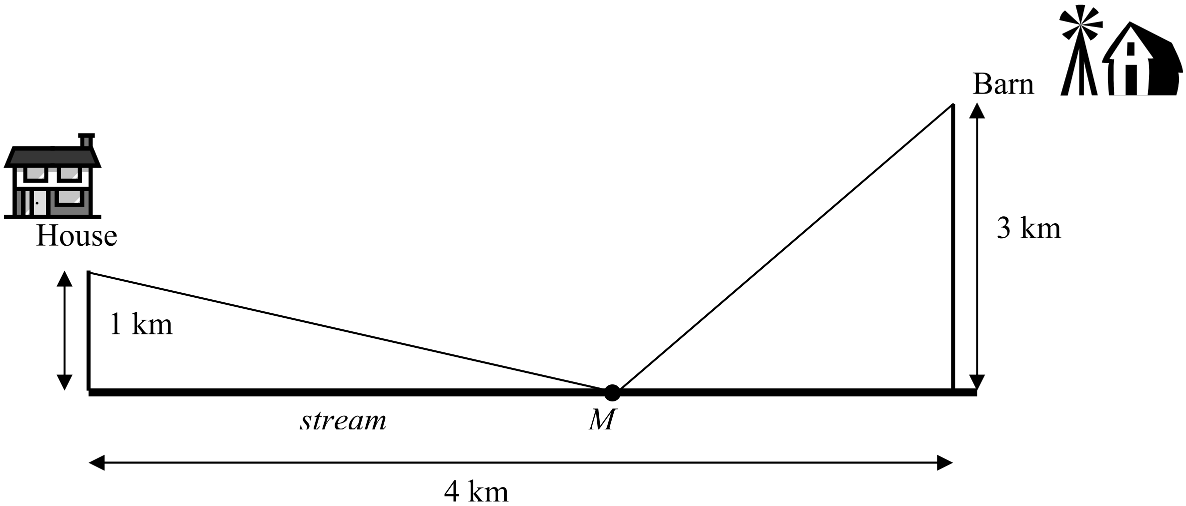
1 + 1 = 2 marks

Total 16 marks

Question 3

Mary, a farmer, needs to collect fresh water from a stream that borders her property. The stream follows a straight line and forms the east–west boundary at the southern end of the property. Mary walks at a constant rate from the house to the stream, fills her pail, and then carries it to the barn to feed the animals.

Initially she walks at a constant speed of 6 km/hr and her path is described in the diagram below.



She walks from the house to a point *M*, 2 km along the stream.

3a. i. Find the distance from the house to the stream (correct to three decimal places) along this path.

Answer

$$d_{HM} = \sqrt{1+4} = \sqrt{5} = 2.236\,\text{km}$$

3a. ii. Find the distance from the stream to the barn (correct to three decimal places) along this path.

Answer

$$d_{HM} = \sqrt{4+9} = \sqrt{13} = 3.606\,\text{km}$$

Marks

- 1 mark for each correct answer (correct to three decimal places)

1 + 1 = 2 marks

3b. Find the total time taken in hours (correct to three decimal places) to complete the task.

Worked solution

$$\begin{aligned} \text{time} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{2.236}{6} + \frac{3.606}{6} \\ &= 0.974\,\text{hours} \end{aligned}$$

Marks

- 1 mark for showing calculation involving speed and distance
- 1 mark for the correct answer (correct to three decimal places)

2 marks

SECTION 2 – continued

Suppose that instead of walking to point M, Mary walks to a point P, x km from the house along the stream.

3c. Show that the time taken, *T(x),* to complete the task in terms of x is given by

$$T(x) = \frac{\sqrt{x^2+1}}{6} + \frac{\sqrt{x^2-8x+25}}{6} \text{ for } x \in [0,4]$$

Answer

Using two right angled triangles gives the distance from the house to the stream as

$\sqrt{1+x^2}$

and the distance from the stream to the barn as

$\sqrt{(4-x)^2+9} = \sqrt{x^2-8x+16+9} = \sqrt{x^2-8x+25}$

Then use $T = \dfrac{\text{distance}}{\text{speed}}$ with the speed at 6km/hr.

Marks

- 1 mark for each of the 2 distances
- 1 mark for giving in the form as stated

2 marks

3d. Find the location of the point on the stream that Mary should walk to in order to complete the task in the shortest time. State the shortest time in hours correct to three decimal places.

Answer

Note that there is no need to use calculus, as the question does not state this is required.

Find the minimum turning point on a calculator as being (1, 0.943). Therefore, Mary should walk to a point that 1 km along the stream to achieve the shortest time of 0.943 hrs.

Marks

- 1 mark for the value of x
- 1 mark for the shortest time

2 marks

Realistically, Mary cannot carry the full pail as quickly as the empty pail.

3e. If her speed with the empty pail is k times her speed of 6 km/hr with the full pail (where $k \geq 1$) , write an expression for the time taken to complete the task in terms of k.

Worked solution

speed with full pail $= 6$

speed with empty pail $= 6k$

$$\text{time} = \frac{\sqrt{x^2+1}}{6k} + \frac{\sqrt{x^2-8x+25}}{6}$$

Marks

- 1 mark for the correct speeds
- 1 mark for the correct expression

2 marks

3f. Show that the quickest path occurs when $k = \dfrac{x}{4-x}\sqrt{\dfrac{(x^2-8x+25)}{(x^2+1)}}$, $x \neq 4$.

Worked solution

Quickest path occurs when $\dfrac{dT}{dx} = 0$

$$\frac{dT}{dx} = \frac{\frac{1}{2}2x(x^2+1)^{\frac{-1}{2}}}{6k} + \frac{\frac{1}{2}(2x-8)(x^2-8x+25)^{\frac{-1}{2}}}{6}$$

$$= \frac{x}{6k\sqrt{x^2+1}} + \frac{x-4}{6\sqrt{x^2-8x+25}}$$

Let $\dfrac{dT}{dx} = 0$

$$\Rightarrow \frac{x}{6k\sqrt{x^2+1}} = -\left(\frac{x-4}{6\sqrt{x^2-8x+25}}\right)$$

$$= \frac{4-x}{6\sqrt{x^2-8x+25}}$$

$$\Rightarrow k = \frac{x}{4-x}\frac{\sqrt{x^2-8x+25}}{\sqrt{x^2+1}} \text{ as required}$$

Marks

- 1 mark for attempting to find derivative
- 1 mark for the correct derivative
- 1 mark for equating to zero
- 1 mark for the correct simplification

4 marks

Total 14 marks

Question 4

Computer modelling for weather forecasting suggests that the probability of rain occurring on a particular day depends strongly on whether it has rained the day before. If it rains on a particular day, the probability of rain on the next day is 0.6. If it doesn't rain on a particular day, then the probability of it not raining the next day is 0.75.

Suppose it rains one Sunday.

4a. i. What is the probability that it will rain on Monday, Tuesday and Wednesday?

Answer

$0.6^3 = 0.216$

Marks

- 1 mark for the correct answer

4a. ii. What is the probability that it will rain on exactly one of the days from Monday to Wednesday?

Worked solution

There are three possibilities. If R signifies 'rain' and N signifies 'no rain',

$\Pr(RNN) = 0.6 \times 0.4 \times 0.75$

$\Pr(NRN) = 0.4 \times 0.25 \times 0.4$

$\Pr(NNR) = 0.4 \times 0.75 \times 0.25$

$\Rightarrow$

$\Pr(RNN) + \Pr(NRN) + \Pr(NNR)$

$= (0.6 \times 0.4 \times 0.75) + (0.4 \times 0.25 \times 0.4) + (0.4 \times 0.75 \times 0.25)$

$= 0.295$

Marks

- 1 mark for recognising three possibilities
- 1 mark for the correct probabilities
- 1 mark for the answer

1 + 3 = 4 marks

When it does rain, the time, in minutes, that the rain falls is described by the random variable with the probability density function

$$f(t) = \begin{cases} kt(100 - t^2) & 0 \le t \le 10 \\ 0 & \text{otherwise} \end{cases}$$

4b. i. Show that the value of k is 0.0004.

Answer

Because curve is a probability density function,
The area under the curve is one and so we have—

$$\int_0^{10} kt(100 - t^2)dt = 1 \Rightarrow k[50t^2 - t^4]_0^{10} = 1$$

$$k((5000 - 2500) - 0) = 1,\ k = 0.0004$$

Marks

- 1 mark for the correct integral
- 1 mark for getting the correct antiderivative and the correct answer

4b. ii. On a day when it does rain, what is the probability, correct to three decimal places, that it rains for at least six minutes?

Worked solution

$$0.0004\int_6^{10} (100t - t^3)dt = 0.0004[50t^2 - \frac{t^4}{4}]_6^{10}$$

$$= 0.0004[(5000 - 2500) - (1800 - 324)] = 0.410$$

Marks

- 1 mark for setting up integral
- 1 mark for answer
- A graphics calculator could be used, but at least one working step must be shown

2 + 2 = 4 marks

Suppose it rains each day for the next week.

4c. What is the probability, correct to three decimal places, that it rains for at least six minutes on at least three out of the next four days?

Worked solution

$X \sim Bi(n = 4, p = 0.410)$

$\Pr(X \ge 3) = 0.191$

Marks

- 1 mark for recognising the binomial and stating the parameters
- 1 mark for the answer

2 marks

4d. On 21% of occasions when it rains, it rains for more than n minutes. Find the value of n, correct to two decimal places.

Worked solution

$$0.0004\int_{n}^{10}(100t - t^3)dt = 0.21$$

$$\Rightarrow [50t^2 - \frac{t^4}{4}]_n^{10} = 525$$

$$\Rightarrow 5000 - 2500 - 50n^2 + \frac{n^4}{4} = 525$$

$\Rightarrow 1975 = 50n^2 - \frac{n^4}{4}$, then use a calculator to get $n = 7.36$

Marks

- 1 mark for the correct integral
- 1 mark for obtaining the equation in terms of n
- 1 mark for the answer

3 marks

Total 13 marks

END OF WORKED SOLUTIONS