CENTRE FOR STRATEGIC EDUCATION
Leading educational thinking and practice

IARTV NO. 27

STUDENT:	TEACHER:

YEAR 12 – OCTOBER 2006

MATHEMATICAL METHODS

Written test 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the test room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one approved graphics calculator (memory **DOES NOT** need to be cleared) and, if desired, one scientific calculator, one bound reference.
- Students are **NOT** permitted to bring into the test room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 22 pages, with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **name** in the space provided above and on the multiple choice answer sheet.
- All written responses must be in English.

At the end of the test
- Place the answer sheet for multiple choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or other electronic communication devices into the test room.

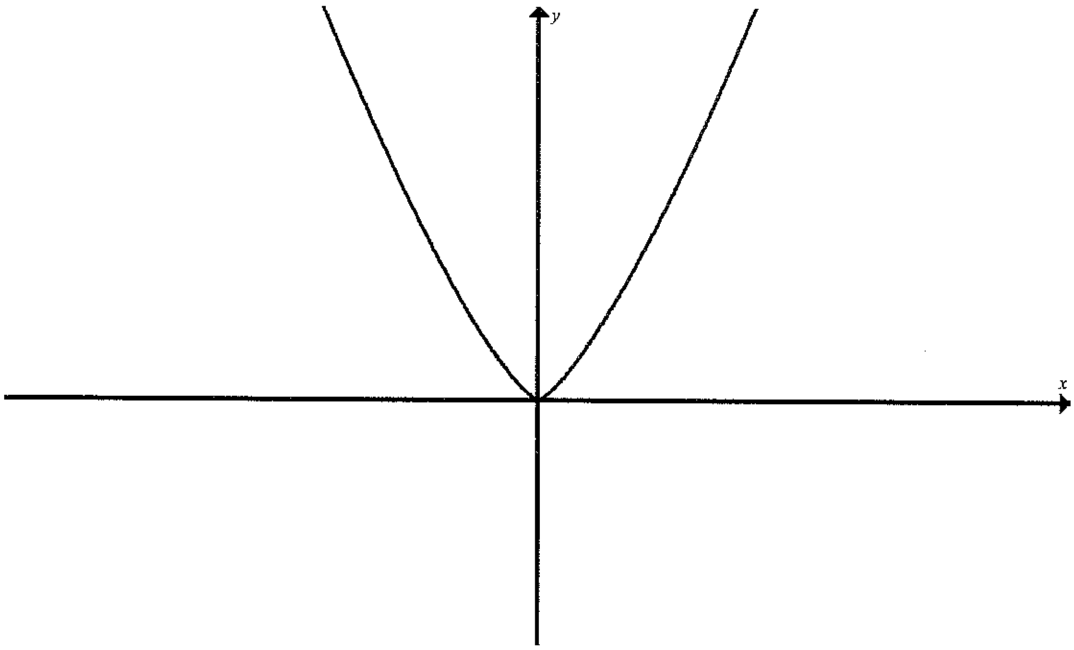
SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheer provided for multiple choice questions.
Choose the response that is correct for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

Out of five equations given I $y=x^{\frac{4}{3}}$, II $y=x^{\frac{5}{3}}$, III $y=x^{\frac{4}{5}}$, IV $y=x^{\frac{5}{4}}$, V $y=x^{\frac{6}{5}}$ the possible equations for the graph shown below are



A. II and IV only
B. I and V only
C. I and III only
D. II and III only
E. I, IV and V only

Question 2

The function $f(x) = |x-3| + |x+2|$ could be expressed as

A. $f(x) = \begin{cases} 2x-1, & x \geq \frac{1}{2} \\ -2x+1, & x < \frac{1}{2} \end{cases}$

B. $f(x) = \begin{cases} -2x+1, & x < -2 \\ 5, & -2 \leq x < 3 \\ 2x-1, & x \geq 3 \end{cases}$

C. $f(x) = \begin{cases} 2x-1, & x \leq -2 \\ -1, & -2 < x \leq 3 \\ -2x+1, & x > 3 \end{cases}$

D. $f(x) = \begin{cases} 2x-1, & x > 3 \\ -2x+1, & x \leq 2 \end{cases}$

E. $f(x) = \begin{cases} 2x-5, & x \geq 3 \\ -1, & x < 3 \end{cases}$

Question 3

Rate of change of $f(x) = 2x^2 - 3x$ at the point $x = 2$ is equal to

A. $\dfrac{2h^2 - 3h}{h}$

B. $\dfrac{2h^2 + 11h}{h}$

C. $\lim\limits_{h \to 0} \dfrac{2h^2 - 3h}{h}$

D. $\lim\limits_{h \to 0} \dfrac{2h^2 + 5h}{h}$

E. $\lim\limits_{h \to 0} \dfrac{2h^2 + 11h}{h}$

Question 4

The inverse of the function $f(x) = -\sqrt{2-x}$ is

A. $f^{-1}(x) = 2 - x^2$, $x \in [0, +\infty)$

B. $f^{-1}(x) = x^2 - 2$, $x \in (-\infty, 2]$

C. $f^{-1}(x) = -(2-x)^2$, $x \in (-\infty, 0]$

D. $f^{-1}(x) = 2 - x^2$, $x \in (-\infty, 0]$

E. $f^{-1}(x) = 2 + x^2$, $x \in [-2, +\infty)$

Question 5

The approximate value of $\log_7 30$ is

A. 1.748

B. 1.477

C. 0.486

D. 0.572

E. 3.401

Question 6

The function $y = x^2 + x$ is first translated 5 units to the right and then dilated from the y – axis by a factor of $\frac{1}{2}$.

The transformed function is

A. $y = (2x-5)^2 + 2x - 5$

B. $y = \left(\frac{1}{2}x - 5\right)^2 + \frac{1}{2}x - 5$

C. $y = (2(x-5))^2 + 2(x-5)$

D. $y = 2\left(x - \frac{5}{2}\right)^2 + 2\left(x - \frac{5}{2}\right)$

E. $y = \left(\frac{1}{2}(x-10)\right)^2 + \frac{1}{2}(x-10)$

Question 7

Solutions to the equation $\sin 2x = \sqrt{3}\,\cos 2x$ in the interval $[\pi,\ 2\pi]$ are

A. $x = \dfrac{7\pi}{6},\ \dfrac{5\pi}{6}$

B. $x = \dfrac{5\pi}{3},\ \dfrac{7\pi}{3}$

C. $x = \dfrac{7\pi}{6},\ \dfrac{5\pi}{3}$

D. $x = \dfrac{5\pi}{6},\ \dfrac{7\pi}{3}$

E. $x = \dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$

Question 8

The derivative $f'(x)$ of $f(x) = |x^2 - 4|$ is

A. $f'(x) = \begin{cases} 2x, & x \in (-\infty,\ -2) \cup (2,\ +\infty) \\ -2x, & x \in [-2,\ 2] \end{cases}$

B. $f'(x) = \begin{cases} 2x, & x \in (-\infty,\ -4) \cup (4,\ +\infty) \\ -2x, & x \in [-4,\ 4] \end{cases}$

C. $f'(x) = \begin{cases} 2x, & x \in (-\infty,\ -2) \cup (2,\ +\infty) \\ -2x, & x \in (-2,\ 2) \end{cases}$

D. $f'(x) = \begin{cases} 2x, & x \geq 2 \\ -2x, & x < -2 \end{cases}$

E. $f'(x) = \begin{cases} 2x, & x^2 - 4 \geq 0 \\ -2x, & x^2 - 4 < 0 \end{cases}$

Question 9

$\frac{d}{dx} \ln |3x^2 - 5|$ is equal to

A. $\frac{6x-5}{|3x^2-5|}$

B. $\frac{1}{|3x^2-5|}$

C. $\frac{6x}{|3x^2-5|}$

D. $\frac{6x}{3x^2-5}$

E. $\left|\frac{6x}{3x^2-5}\right|$

Question 10

The derivative of $e^{\sqrt{\frac{x}{2}}}$ is equal to

A. $\frac{e^{\sqrt{\frac{x}{2}}}}{\sqrt{2x}}$

B. $\frac{e^{\sqrt{\frac{x}{2}}}}{2\sqrt{x}}$

C. $\frac{2e^{\sqrt{\frac{x}{2}}}}{\sqrt{\frac{x}{2}}}$

D. $\frac{e^{\sqrt{\frac{x}{2}}}}{2\sqrt{\frac{x}{2}}}$

E. $\frac{e^{\sqrt{\frac{x}{2}}}}{2\sqrt{2x}}$

Question 11

The equation of the normal to the curve $y = \tan(2x)$ at the point $x = -\frac{\pi}{8}$ is

A. $y+1 = -\frac{1}{4}\left(x+\frac{\pi}{8}\right)$

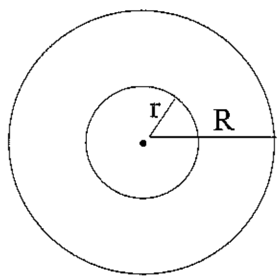
B. $y+1 = 4\left(x+\frac{\pi}{8}\right)$

C. $y+1 = -\frac{1}{2}\left(x+\frac{\pi}{8}\right)$

D. $y+1 = 2\left(x+\frac{\pi}{8}\right)$

E. $y-1 = -\frac{1}{2}\left(x-\frac{\pi}{8}\right)$

Question 12



An annulus is the shape between two circles with the same centre.

The area, A, of an annulus is given by $A = \pi\left(R^2 - r^2\right)$, where R is the radius of the outside circle and r the inside one. If R is fixed while r increases from 3 to 3.1 cm, then the approximate change of the area of the annulus in terms of π is

A. $0.6\pi\ cm^2$

B. $\sqrt{91}\ \pi\ cm^2$

C. $-3.1\pi\ cm^2$

D. $7\pi\ cm^2$

E. $-0.6\pi\ cm^2$

Question 13

Sand is poured into a heap in the shape of a cone with the angle $120°$ at the vertex. If the height of the heap is increasing at 3 cm per second when the height is 7 cm, then the volume of the cone is increasing at

A. $90\pi^2\ cm^3/\text{sec}$

B. $441\pi\ cm^3/\text{sec}$

C. $602\pi\ cm^3/\text{sec}$

D. $216\pi\ cm^3/\text{sec}$

E. $380\pi\ cm^3/\text{sec}$

Question 14

The integral $\int \frac{x}{x+3}\,dx$ is equal to

A. $\ln|x+3|+c$

B. $\frac{x^2}{2}\ln|x+3|+c$

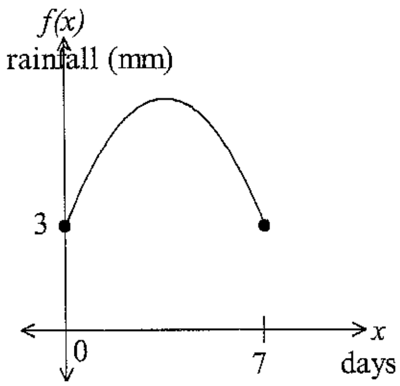
C. $x-3\ln|x+3|+c$

D. $x+\frac{x^2}{6}+c$

E. $x+\frac{1}{3}+c$

Question 15

The amount of rainfall (mm) on a tropical island over a week can be modelled by a function $f(x)=-0.3x(x-7)+3$ for $0\le x\le 7$.



The average rainfall over 7 days is

A. 3

B. 3.5

C. 4.43

D. 5.45

E. 6.67

Question 16

Given that $a^x=e^{x\ln a}$ the integral $\int 5^{2x}\,dx$ is equal to

A. $2\ln 5\times 5^{2x}+c$

B. $\frac{\ln 5}{2}\times 5^{2x}+c$

C. $2\ln 5\times e^{2x\ln 5}+c$

D. $\frac{2}{\ln 5}\times e^{2x\ln 5}+c$

E. $\frac{1}{2\ln 5}\times 5^{2x}+c$

Question 17

For the function $y=\frac{1}{2}x^4$ the approximate value of the area between the graph and the x – axis using the right endpoint estimate with 4 equal intervals between -2 and 2 is equal to

A. 9.5

B. 9

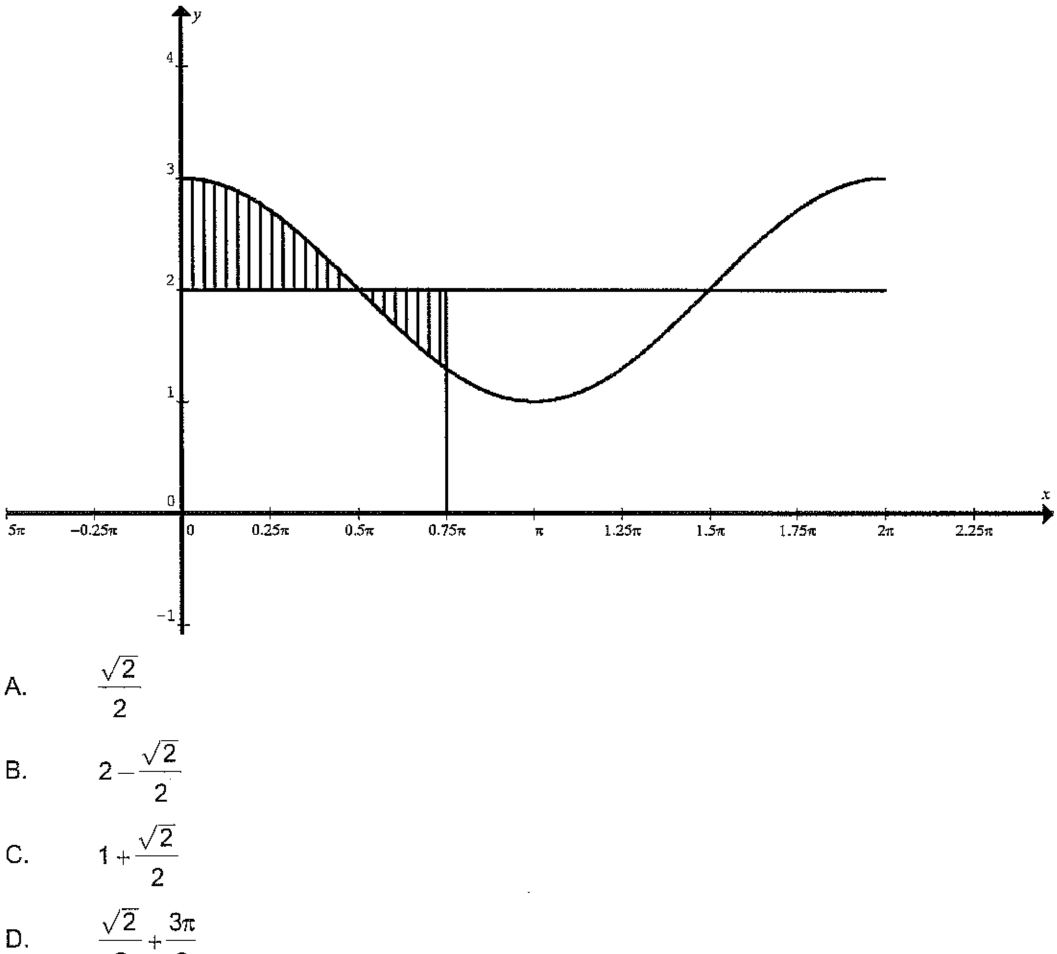
C. 5

D. 5.5

E. 17

Question 18

The area (shaded) bounded by the curve $y=\cos x+2$, the y – axis and the lines $x=\frac{3\pi}{4}$ and $y=2$ is equal to



A. $\frac{\sqrt{2}}{2}$

B. $2-\frac{\sqrt{2}}{2}$

C. $1+\frac{\sqrt{2}}{2}$

D. $\frac{\sqrt{2}}{2}+\frac{3\pi}{2}$

E. $\frac{\sqrt{2}}{2}-\frac{3\pi}{2}$

Question 19

Out of people attending a local gym, 10% are underweight, 20% are overweight, 15% are obese, and the rest have normal weight. Underweight people on average weigh 3kg less than their ideal weight, people within the normal weight range weigh on average 4kg more than the ideal weight, and overweight and obese people have 12 and 18kg extra weight respectively. The standard deviation of the extra weight of the patrons of the gym is

A. 87.1

B. 7.0

C. 6.17

D. 85.3

E. 9.24

Question 20

7% of the people who buy airline tickets cancel their booking. The probability that out of 10 people more than 3 will cancel their booking is

A. 0.972

B. 0.0036

C. 0.0248

D. 0.0283

E. 0.996

Question 21

If the continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 0.1e^{-0.1x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

then $Pr(X > 5)$ is equal to

A. $e^{-0.5}$

B. 0.59

C. 1

D. ∞

E. cannot be determined as $f(x)$ is not a pdf

Question 22

X is a normally distributed random variable with mean equal to 12. If $Pr(X < 10 | X < 12) = 0.2$ then the variance of this distribution is equal to

A. 1.56

B. 2.45

C. 2.38

D. 5.65

E. none of the above

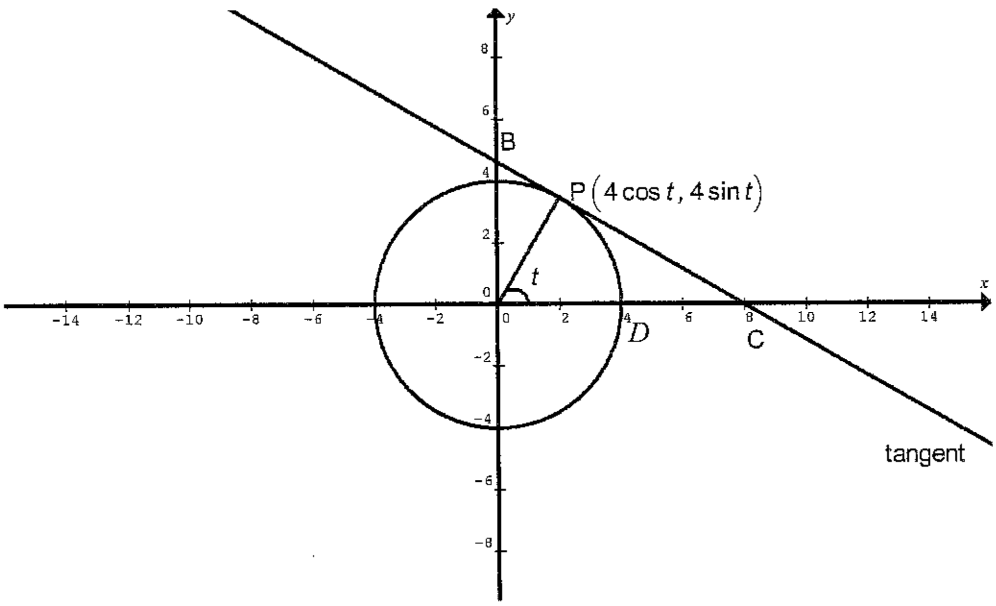
END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Where an instruction to use calculus is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1



In the graph above P is a point on the circle which is centred at the origin and has radius 4 units, O is the origin, and D(4,0). When the angle POD is t radians, then the coordinates of P are $(4\cos t, 4\sin t)$.

a. Find expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

2 marks

b. Using the above results or otherwise, show that $\frac{dy}{dx} = -\cot t$.

1 mark

c. Give an exact value for the gradient of the tangent at P when $t = \frac{5\pi}{6}$.

1 mark

d. State the exact values of t, $0 \le t \le 2\pi$, when the gradient of the tangent is -1.

2 marks

e. Show that the equation of the tangent through P, for a fixed value of *t*, can be written as

$$y = -(\cot t)x + \frac{4}{\sin t}.$$

2 marks

The remaining questions restrict P to the first quadrant. The tangent at P crosses the x and y axes respectively at C and B.

f. Find the coordinates of C and B (in terms of t).

2 marks

g. Using the results of **f** show that the area A of the triangle BOC can be written as

$$A = \frac{8}{\cos t \sin t}.$$

1 mark

The above expression for the area A can be simplified to $A = \frac{4}{\sin 2t}$.

h. Give an expression for $\frac{dA}{dt}$.

1 mark

i. Use Calculus to find the minimum area for the triangle BOC if $0 \le t \le \frac{\pi}{2}$ (you should justify that it is a minimum value).

2 marks

Total 14 marks

Question 2

In the eighteenth century a Scottish mathematician, Colin Maclaurin found the following polynomial expression for the trigonometric function $\sin x$:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$$

Consider the cubic approximation :

$\sin x \approx x - \frac{x^3}{6}$ where $\approx$ means ' is approximately equal to '.

Let $P(x) = x - \frac{x^3}{6}$ where $-\pi \le x \le \pi$ and $S(x) = \sin x$ where $-\pi \le x \le \pi$.

a. Factorise $P(x)$ and find the exact values of the x-intercepts for $y = P(x)$.

2 marks

b. Find $P'(x)$ (i.e. $\frac{dP}{dx}$).

1 mark

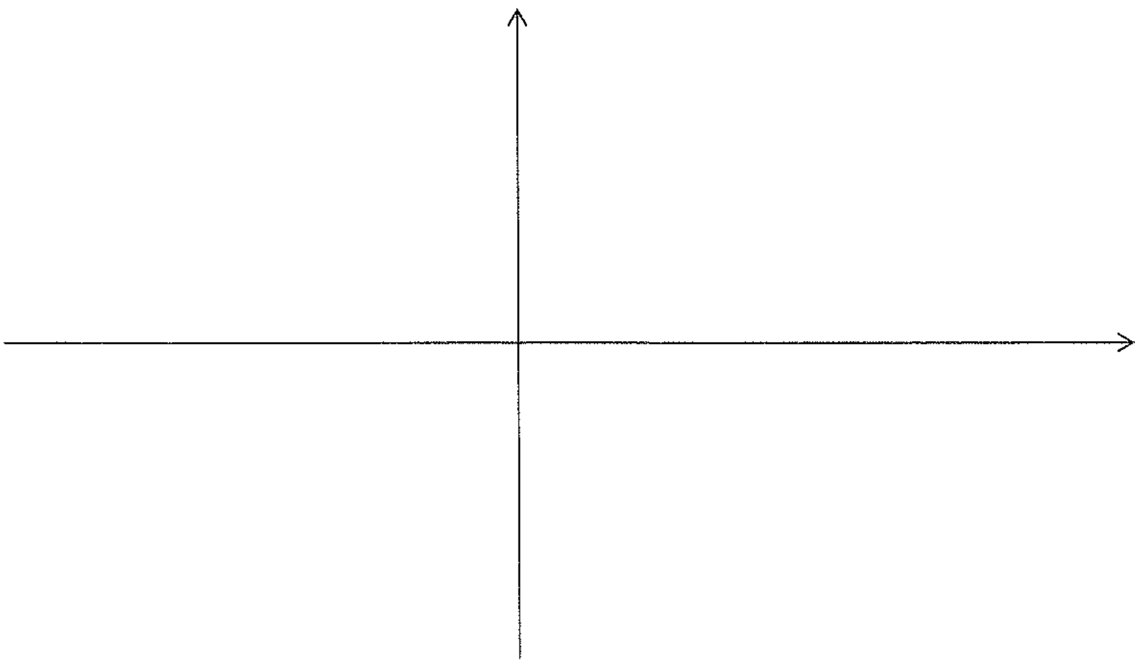
c. Give the exact coordinates of the turning points of $y = P(x)$.

2 marks

It is difficult to see the fine detail of the graphs of $y = P(x)$ and $y = S(x)$ using the graphics calculator. The following information should assist you with the graphing:

$$S(x) > P(x) \text{ for } 0 < x < \pi$$
$$S(x) < P(x) \text{ for } -\pi < x < 0.$$

d. On the axes provided below sketch the graphs of $y = S(x)$ and $y = P(x)$ (endpoints should be included and each graph labelled).



3 marks

e. On the previous graph shade the area between $y = P(x)$ and $y = S(x)$.

1 mark

f. Use calculus to obtain an exact value for the area between $y = P(x)$ and $y = S(x)$.

3 marks

We define the error of the cubic approximation of $S(x) = \sin x$ as the difference between $P(x)$ and $S(x)$.

g. State the exact error when $x = \frac{\pi}{4}$.

2 marks

Further, we define the total error on an interval as the sum of all the errors on that interval.

h. Calculate the total exact error for $-\pi \le x \le \pi$.

1 mark

Total 15 marks

Question 3

In this problem give all answers to 4 decimal places unless otherwise stated.

The Bright Light Company produces electric light globes. One of its popular globes is the 60W standard model. After testing a large number of these standard globes, it was found that the lifetime of these globes was normally distributed with a mean life of 100 hours and a standard deviation of 10 hours.

a. What is the probability that a randomly selected standard globe will have a life of more than 105 hours?

1 mark

b. State the probability that a randomly selected standard globe will have a life between 87 and 115 hours.

2 marks

c. Give the number of hours, to the nearest hour, that will be exceeded by 10% of the standard globes.

2 marks

d. Find the time interval that you would expect the lifetime of 95% of the globes to be within.

1 mark

The Company packages the standard model globes in boxes of 12 globes.

e. Calculate the probability that a box of standard globes contains exactly 2 globes which last less than 90 hours.

3 marks

As a guarantee of its quality, the company offers to replace any ' Unsatisfactory Box ' which contains more than two globes which last less than 90 hours.

f. State the probability of an ' Unsatisfactory Box '.

1 mark

The profits for the boxes of standard globes are :

' Satisfactory Box '	\$4 profit per box
' Unsatisfactory Box '	\$2 loss per box

g. Find the expected profit and variance for a randomly chosen box of standard globes.

2 marks

h. If the company sells 1000 boxes of these standard globes, what is the expected profit?

1 mark

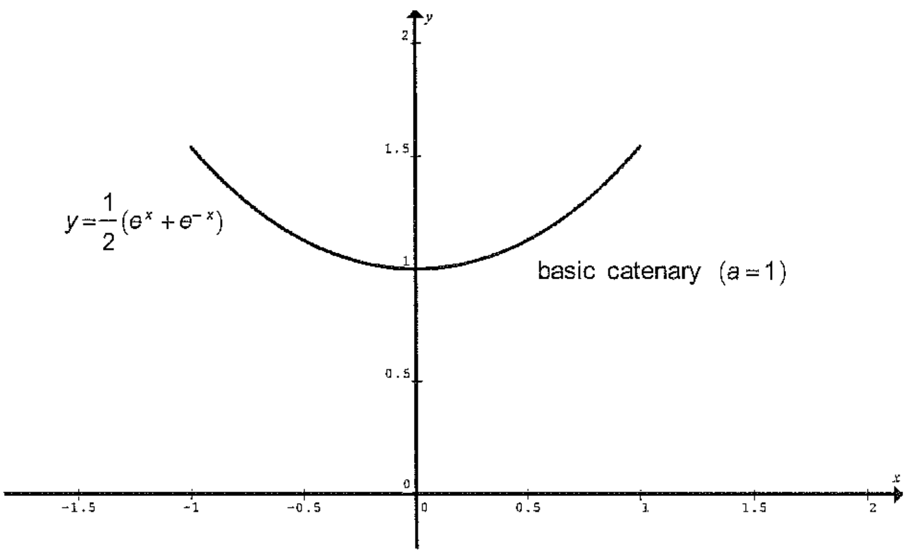
Total 13 marks

Question 4

In Mathematics, the **catenary** is the shape of a hanging flexible chain when supported at its ends. The general equation of the shape of the catenary is given by:

$$y = \frac{a}{2}(e^{\frac{x}{a}} + e^{\frac{-x}{a}})$$ where *a* is constant.

The graph shows the case where $a = 1 \text{ and } -1 \le x \le 1$ and will be described as the *basic catenary graph.*



a. Give the exact coordinates of the endpoints for the basic catenary.

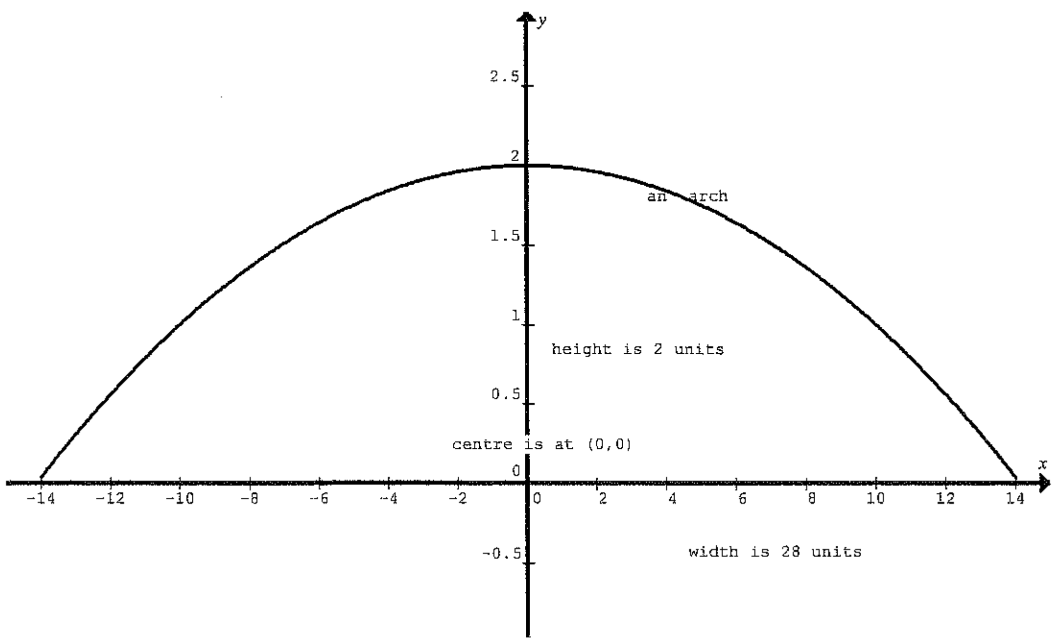
1 mark

An *inverted catenary or catenary arch* is formed when the catenary is reflected in the *x*-axis.

b. For the basic catenary given in the above graph, state the equation of the corresponding basic catenary arch.

1 mark

The basic catenary arch can be transformed to create arches of different heights and widths.



The arch shown above is centred at the origin, has a height of 2 units and a width of 28 units.

The Gateway Arch in St. Louis, Missouri has its equation displayed inside the arch:

$$y = 757.7 - 127.7(e^{\frac{x}{127}} + e^{-\frac{x}{127}})$$

where x is the horizontal displacement from the centre of the arch in feet, and

y is the height above the centre of the arch in feet.

c. Find the height of the St. Loius arch to the nearest foot.

1 mark

The Spanish architect Antoni Gaudi made extensive use of catenary arches in his cathedral Sagrada Familia. He would build inverted scale models of the arches that he wished to construct by hanging weights from strings.

d. Briefly explain the transformations that will take the basic catenary $y = \frac{1}{2}(e^{x} + e^{-x})$ to the catenary arch:

$$y = -\frac{1}{2}(e^{\frac{x}{w}} + e^{\frac{-x}{w}}), \quad -w \le x \le w.$$

1 mark

e. Show that the catenary arch in **d** has a height of $\frac{(e-1)^2}{2e}$.

2 marks

An arch can be modelled by the equation:

$$y = 4 - \frac{e^{x} + e^{-x}}{2}, \ -1 \le x \le 1$$

Where *y* is the height of the arch above the floor in metres, and *x* is the horizontal displacement from the centre in metres.

f. State the exact values of *x* where the height of the arch is 2 metres above the floor.

4 marks

The general equation of an arch can be written as:

$$y = b - \frac{a}{2}(e^{\frac{x}{a}} + e^{\frac{-x}{a}}), \ -c \le x \le c \text{ where } a, b, c \text{ are constants.}$$

g. If $y \ge 0$ on its domain, find and simplify an expression for the area between this curve and the x-axis and bounded by $x = -c$ and $x = c$.

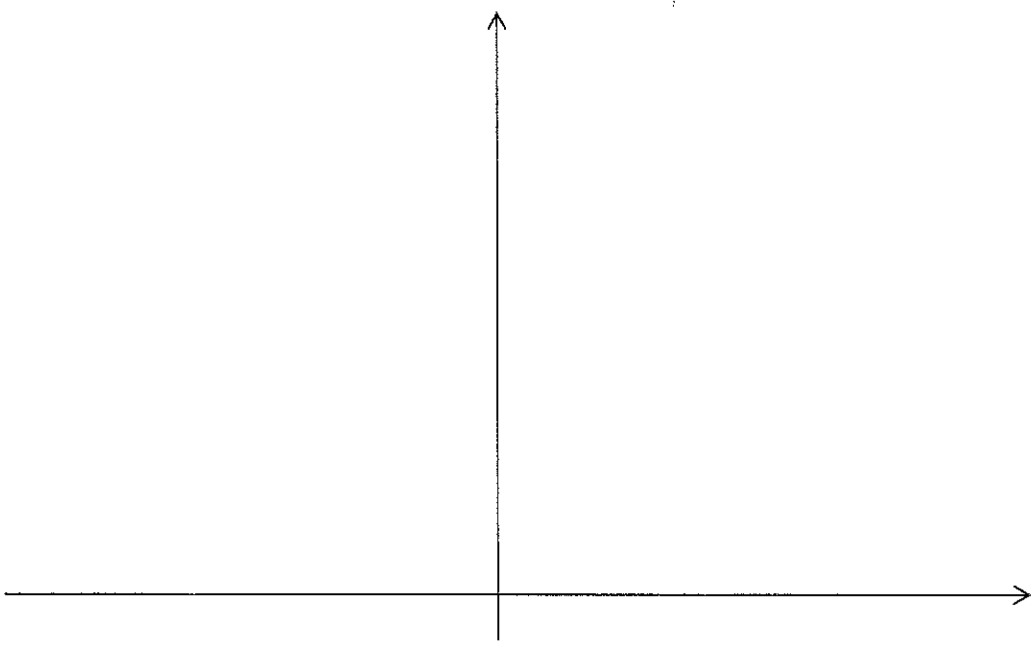
3 marks

h. Use your graphics calculator to obtain sketches of the following arches :

$$y = 28.413 - 9.207(e^{x} + e^{-x})$$

$$y = 30.413 - 9.207(e^{x} + e^{-x}) .$$

Sketch on the axes below showing axes intercepts and ensuring that the y values are non-negative for each arch.



3 marks

Total 16 marks

END OF QUESTION AND ANSWER BOOK