VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2005

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

									Letter
Figures									
Words									

MATHEMATICAL METHODS (CAS)

PILOT STUDY

Written examination 2

(Analysis task)

Monday 7 November 2005

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.45 am (1 hour 30 minutes)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
4	4	55

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For the TI-92, Voyage 200 or approved computer based CAS, their full functionality may be used, but other programs or files are not permitted.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 12 pages, with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions

- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

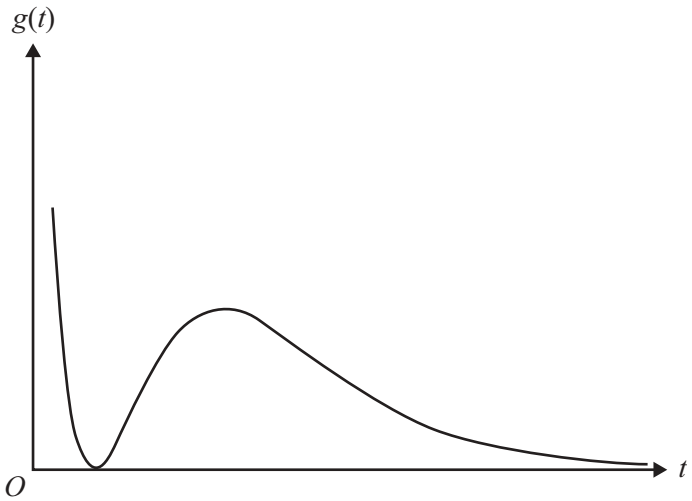
Let $f: [0, \infty) \rightarrow R, f(t) = 2e^{-t}$.

a. **i.** State the range of f.

ii. Find the rule for the inverse of f and state its domain.

1 + 2 = 3 marks

b. Let $g: [0, \infty) \rightarrow R, g(t) = (t - 1)^2 e^{-t}$.
Part of the graph of g is shown.



i. The rule for the derivative of g may be expressed in the form $g'(t) = (-t^2 + bt + c)e^{-t}$.
Find the exact values of b and c.

ii. The graph of $y = g(t)$ has stationary points $(1, p)$ and (m, n).
Find the exact values of p, m and n.

iii. For the function $q: [0, \infty) \rightarrow R$, with rule $q(t) = 2g(t) - 5$, state the exact coordinates of the stationary points of the graph of $y = q(t)$.

3 + 2 + 2 = 7 marks

Question 1 – continued
TURN OVER

c. The function $h: R \rightarrow R$, $h(t) = (t^2 + at + 10)e^{-t}$, where a is a real constant, has derivative $h'(t) = (-t^2 + (2 - a)t + (a - 10))e^{-t}$.

Find the values of a such that

i. the graph of $y = h(t)$ has exactly one stationary point

ii. $h'(t) < 0$ for all $t \in R$.

3 + 2 = 5 marks

Total 15 marks

Question 2

Oz-Online is an Internet provider company, with more than one million customers, and has two Internet access plans.

- The Dial-up Plan
- The Cable-modem Plan

80% of customers have the Dial-up Plan and 20% of customers the Cable-modem Plan.

a. If 10 customers of Oz-Online are selected at random

i. what is the probability, correct to four decimal places, that exactly eight customers are on the Dial-up Plan

ii. how many customers would be expected to be on the Dial-up Plan?

2 + 1 = 3 marks

b. The marketing manager for Oz-Online has decided that the monthly hours used by Dial-up Plan customers is a random variable with probability density function given by

$$f(t) = \begin{cases} 0.05e^{-0.05t} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

i. Find the mean number of hours used in a month by a randomly selected Dial-up Plan customer.

ii. What percentage, correct to the nearest per cent, of Dial-up Plan customers uses more than 30 hours in a month?

iii. What is the probability, correct to three decimal places, that a randomly chosen Dial-up Plan customer uses more than 40 hours per month, given that the customer uses more than 30 hours per month?

2 + 2 + 2 = 6 marks

c. To try to increase the percentage of customers using a cable modem, Oz-Online offers all its **present** customers the Cable-modem Plan with unlimited hours, and at a reduced cost.

The marketing manager estimates that each month 20% of the customers on the Dial-up Plan will change to the Cable-modem Plan, and 5% of the customers on the Cable-modem Plan will switch to the Dial-up Plan. No **present** customers stop using Oz-Online.

i. After four months what proportion of the **present** customers, correct to two decimal places, will now be on the Dial-up Plan?

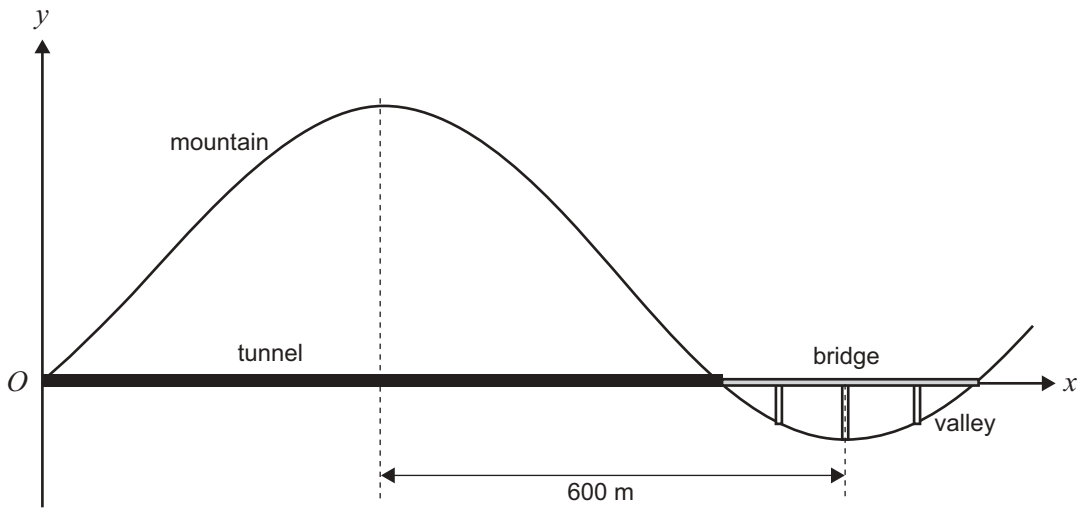
ii. How many months would this change pattern need to continue for the proportion of customers on the Dial-up Plan to be less than 0.25?

3 + 2 = 5 marks

Total 14 marks

Question 3

A hydroelectric authority is proposing to build a horizontal pipeline which will pass through a new tunnel and over a bridge. The diagram below shows a cross-section of the proposed route with a tunnel through the mountain and a bridge over the valley to carry the pipeline.



The boundary of the cross-section can be modelled by a function of the form

$$y = 100\cos\left(\frac{\pi(x-400)}{600}\right) + 50\,, \quad 0 \le x \le 1600$$

where y is the height, in metres, above the proposed bridge and x is the distance, in metres, from a point O where the tunnel will start.

a. What is the height (in metres) of the top of the mountain above the bridge?

1 mark

b. How many metres below the bridge is the bottom of the valley?

1 mark

c. What is the exact length of

i. the tunnel

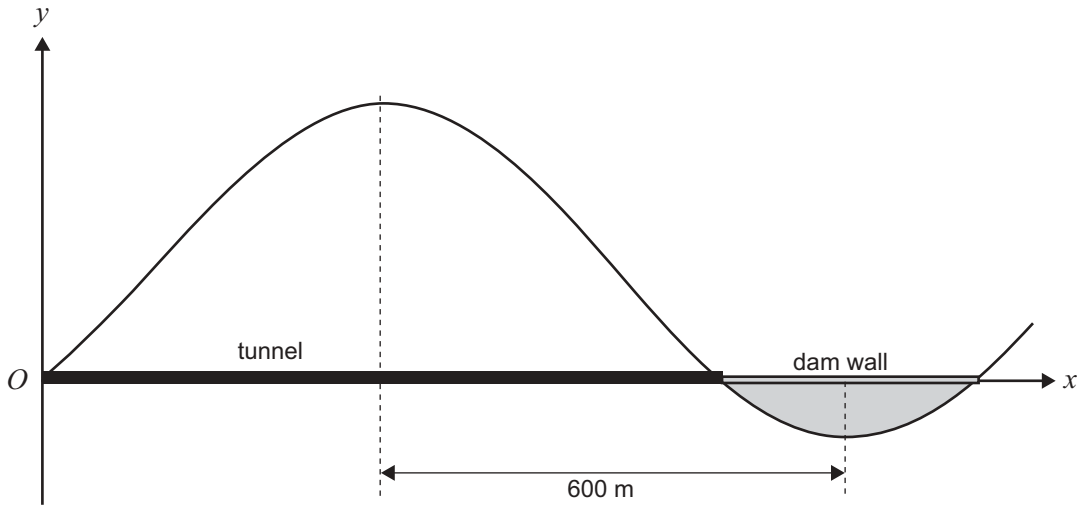
ii. the bridge?

1 + 1 = 2 marks

d. What would be the length (correct to the nearest metre) of the **tunnel** if it were built 20 m higher up the mountain?

2 marks

A second proposal is to build a solid concrete dam instead of a bridge. The shaded area in the diagram below also shows a cross-section of the dam wall.

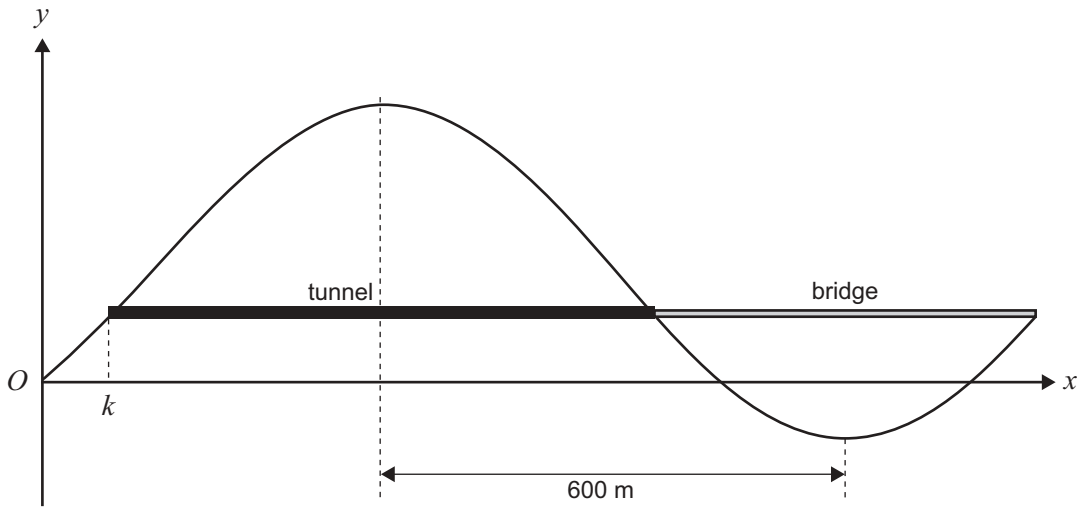


e. **i.** Write a definite integral, the value of which is the area of the cross-section of the dam wall.

ii. Find the area of the cross-section of the dam wall, correct to the nearest square metre.

2 + 1 = 3 marks

A third proposal is to build the tunnel and bridge above the original proposed position.



f. Suppose the tunnel is built at a height such that it starts at a point on the mountain when $x = k$, $0 < k < 400$.

i. Find the length of the tunnel in terms of k.

ii. Find the length of the bridge in terms of k.

iii. The estimated total cost, C thousand dollars, of building the tunnel and bridge for this third proposal is equal to the sum of the square of the length (in metres) of the tunnel and the square of the length (in metres) of the bridge.

Write down an expression for the estimated total cost of building the tunnel and the bridge if the tunnel starts when $x = k$, in terms of k.

iv. Hence find the exact value of k for which the estimated cost of the proposal is minimum.

1 + 1 + 1 + 2 = 5 marks

Total 14 marks

Question 4

The pollution level, y units, along a straight road between two factories, A and B, which are 10 km apart, is given by

$$y = \frac{p}{x+1} + \frac{q}{11-x}, \text{ where } 0 \leq x \leq 10$$

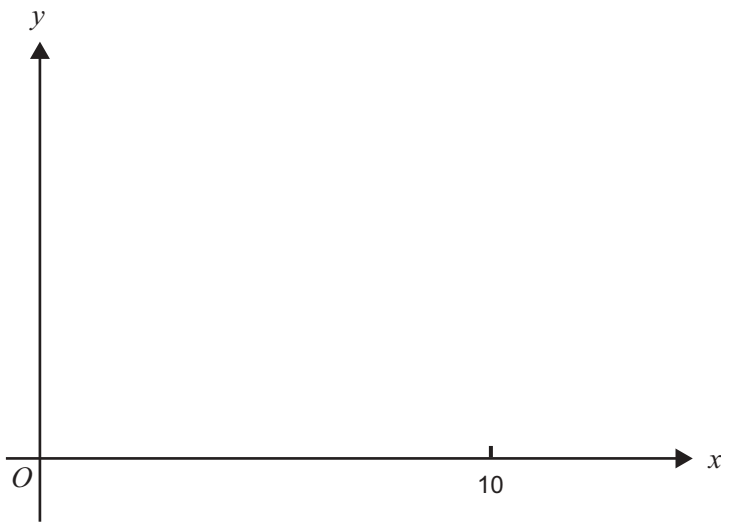
where x km is the distance from Factory A, and p and q are positive constants.

a. On a particular day the values of p and q are measured to be $p = 9$ and $q = 4$.

i. Find $\frac{dy}{dx}$

ii. Find the exact value of x at which the pollution level y is a minimum and the exact value of this minimum.

iii. Sketch the graph of $y = \frac{9}{x+1} + \frac{4}{11-x}$, where $0 \leq x \leq 10$, on the axes below. Clearly label the endpoints and turning point with their coordinates. Exact values are required.



iv. Jack travels from Factory A to Factory B along the road. For what length of his journey (in kilometres correct to three decimal places) is the pollution level less than 5?

1 + 3 + 2 + 1 = 7 marks

b. On another day only the value of q is known, $q = 4$. The pollution level, y units, is given by

$$y = \frac{p}{x+1} + \frac{4}{11-x}, \text{ where } 0 \leq x \leq 10 \text{ and } p \text{ is a positive constant.}$$

For what values of p does

i. the maximum value of y occur when $x = 10$

ii. the minimum value of y occur when $x = 10$?

3 + 2 = 5 marks

Total 12 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS (CAS)
PILOT STUDY

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

Mathematical Methods CAS Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

volume of a pyramid: $\frac{1}{3}Ah$

curved surface area of a cylinder: $2\pi rh$

volume of a sphere: $\frac{4}{3}\pi r^3$

volume of a cylinder: $\pi r^2 h$

area of a triangle: $\frac{1}{2}bc\sin A$

volume of a cone: $\frac{1}{3}\pi r^2 h$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax}dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x}dx = \log_e|x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

average value: $\frac{1}{b-a}\int_a^b f(x)dx$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Discrete distributions			
	$\Pr(X = x)$	mean	variance
general	$p(x)$	$\mu = \sum x\,p(x)$	$\sigma^2 = \sum (x-\mu)^2 p(x)$ $= \sum x^2 p(x) - \mu^2$
binomial	${}^nC_x\, p^x (1-p)^{n-x}$	np	$np(1-p)$
hypergeometric	$\frac{{}^DC_x\ {}^{N-D}C_{n-x}}{{}^NC_n}$	$n\frac{D}{N}$	$n\frac{D}{N}\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right)$
Continuous distributions			
	$\Pr(a < X < b)$	mean	variance
general	$\int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\,f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$ $= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$
normal	If X is distributed $\mathrm{N}(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, then Z is distributed $\mathrm{N}(0, 1)$, $f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$		

Table 1 Normal distribution – cdf

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	4	8	12	16	20	24	28	32	36
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	4	8	12	16	20	24	28	32	35
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	4	8	12	15	19	23	27	31	35
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	4	8	11	15	19	23	26	30	34
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	4	7	11	14	18	22	25	29	32
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	3	7	10	14	17	21	24	27	31
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	3	6	10	13	16	19	23	26	29
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7793	.7823	.7852	3	6	9	12	15	18	21	24	27
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	3	6	8	11	14	17	19	22	25
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	3	5	8	10	13	15	18	20	23
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	2	5	7	9	12	14	16	18	21
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830	2	4	6	8	10	12	14	16	19
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	2	4	6	7	9	11	13	15	16
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	2	3	5	6	8	10	11	13	14
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1	3	4	6	7	8	10	11	13
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1	2	4	5	6	7	8	10	11
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1	2	3	4	5	6	7	8	9
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1	2	3	3	4	5	6	7	8
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1	1	2	3	4	4	5	6	6
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1	1	2	2	3	4	4	5	5
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	0	1	1	2	2	3	3	4	4
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	0	1	1	2	2	2	3	3	4
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	0	1	1	1	2	2	2	3	3
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	0	1	1	1	1	2	2	2	2
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	0	0	1	1	1	1	1	2	2
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	0	0	0	1	1	1	1	1	1
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	0	0	0	0	1	1	1	1	1
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	0	0	0	0	0	1	1	1	1
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	0	0	0	0	0	0	0	1	1
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	0	0	0	0	0	0	0	0	0
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	0	0	0	0	0	0	0	0	0
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	0	0	0	0	0	0	0	0	0
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	0	0	0	0	0	0	0	0	0
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	0	0	0	0	0	0	0	0	0
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	0	0	0	0	0	0	0	0	0
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	0	0	0	0	0	0	0	0	0
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0

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