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MATHMATICAL METHODS

WRITTEN EXAMINATION 2 - SOLUTIONS

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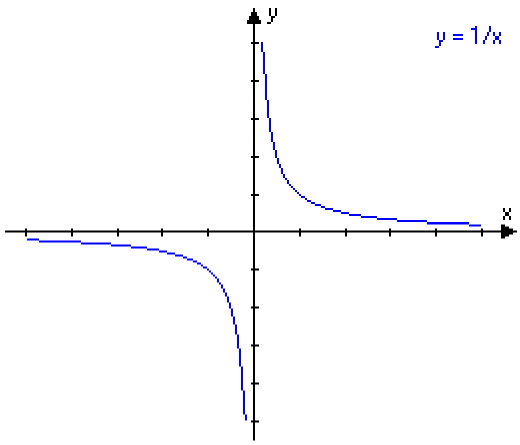
Written Examination 2: Analysis Task Suggested Solutions

Question 1

a. The function $f(x) = \frac{24}{x+3} - 6$ does not allow $x = -3$. Thus the largest domain is R\{-3}.

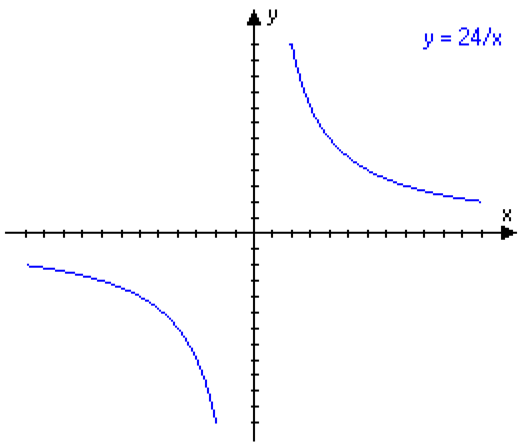
1 mark

b. Applying following transformations for the graph $y = \frac{1}{x}$ we can obtain the graph of $f(x) = \frac{24}{x+3} - 6$



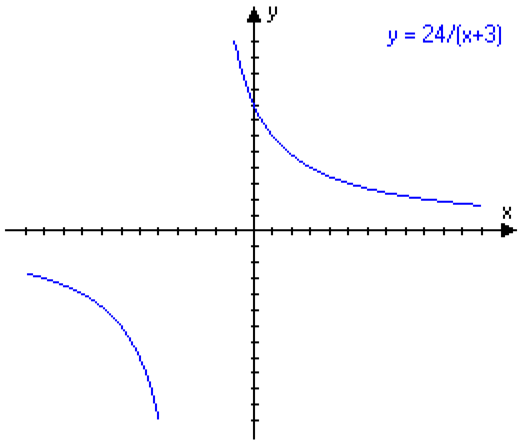
i. First, dilate by a factor of 24 in a vertical direction away from the x-axis.

1 mark



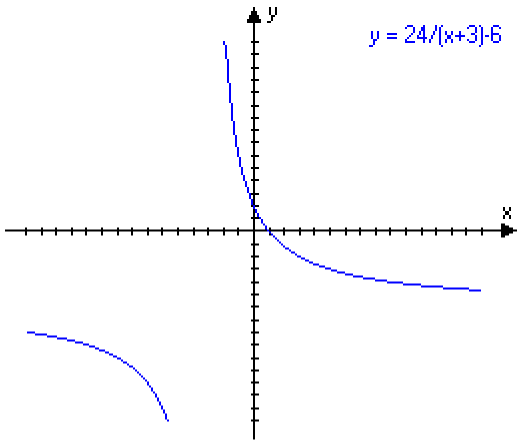
ii. Second translate the graph to the left (in a negative direction) along the x-axis by 3 units.

1 mark



iii. Third, translate the graph down (i.e. in a negative y-direction) by 6 units.

1 mark



c. When $x=0 \Rightarrow f(0)=\frac{24}{3}-6=2$ and when $f(x)=0 \Rightarrow \frac{24}{x+3}-6=0 \Rightarrow x=1$

$\therefore$ Graph passes through the point (0,2) and (1,0).

1 mark

d. **i.**

$f(x)=\frac{24}{x+3}-6$

$g:(-3,\infty)\to R$ such that $g(x)=f(x)$

$\therefore$ domain of $g(x)$ is (-3,∞) and

range of $g(x)$ is (-6,∞)

1 mark

ii.

If $y=g(x)=\frac{24}{x+3}-6$ then inverse function of $g(x)$ is given by

$x=\frac{24}{y+3}-6 \quad \Rightarrow y+3=\frac{24}{x+6}$

$\therefore g^{-1}(x)=y=\frac{24}{x+6}-3$

2 marks

iii When $x=0 \Rightarrow g^{-1}(0)=\frac{24}{6}-3=1$ and when

$f(x)=0 \Rightarrow \frac{24}{x+6}-3=0 \Rightarrow x=2$

$\therefore$ Graph passes through the point (0,1) and (2,0).

1 mark

iv. Domain of g^{-1} is the range of g.

$\therefore$ the domain of g^{-1} is (-6, ∞). **1 mark**

v. Solutions of $g(x)=x$ is given by

$$\frac{24}{x+3}-6=x$$

$$\therefore (x+3)(x+6)=24$$

$$\Rightarrow x^2+9x-6=0$$

$$\therefore x=\frac{-9\pm\sqrt{105}}{2} \Rightarrow x=-9.62 \ or \ 0.62$$

1 mark

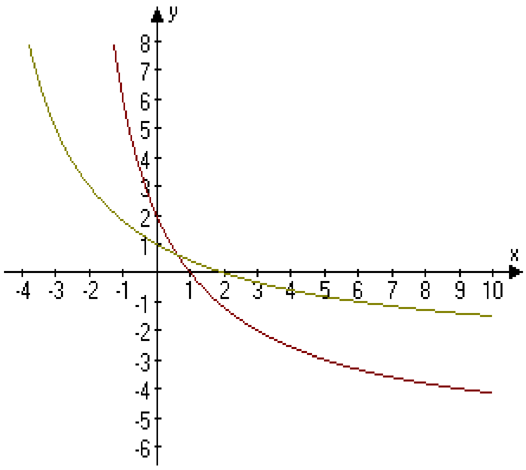
$\therefore$ Solution in the domain of g is 0.62

1

mark

$g(x),\ g^{-1}$ take the same value on the line $y = x$. Hence $g(x) = g^{-1}(x) = x$ is satisfied when $x = 0.62$. **1 mark**

vi.



2 marks

Question 2

$$P(t) = 1500e^{-0.05t}$$

a. When $t = 0,\ P(0) = 1500$ **1 mark**

b. When $P(t) = 750$.

$$750 = 1500e^{-0.05t}$$

$$\therefore e^{0.05t} = 2$$

$$0.05t = \log_e 2$$

$$t = \frac{\log_e 2}{0.05} = 13.86 \approx 14 \text{ weeks}$$

1 mark

c. $\dfrac{dP}{dt} = 1500 \times (-0.05)e^{-0.05t} = -75e^{-0.05t}$

1 mark

d. i. When $t = 2$

$$\left(\frac{dP}{dt}\right)_{t=2} = -75e^{-0.1} = -67.86$$

$$\approx -68 \text{ rabbits/week}$$

1 mark

ii. When $t=10$

$$\left(\frac{dP}{dt}\right)_{t=10}=-75e^{-0.5}=-45.49$$
$$\approx -46 \text{ rabbits/week}$$

1 mark

e. Domain is $[16,\infty)$. **1 mark**

f. $P(t)=P_0+12(t-16)\log_e(2t-31)$

when $t=16$ $P(16)=P_0=1500e^{-0.05\times 16}$ ($\because$ Part (a))

$$=673.99\approx 674$$
$$\therefore P(t)=674+12(t-16)\log_e(2t-31)$$

1 mark

g. When $t=32$

$$P(32)=674+12(16)\log_e(33)$$
$$=1345.33\approx 1346$$

1 mark

h. $$\frac{dP}{dt}=\frac{24(t-16)}{(2t-31)}+12\log_e(2t-31)$$

2 marks

i. **i.** When $t=20$

$$\left(\frac{dP}{dt}\right)_{t=20}=\frac{24\times 4}{9}+12\log_e 9$$
$$=\frac{32}{3}+12\times 2.1972$$
$$=10.6667+26.3667=37.0334$$
$$\approx 37 \text{ rabbits/week}$$

2 marks

ii. When $t=32$ $\left(\frac{dP}{dt}\right)_{t=32}=\frac{24\times 16}{33}+12\log_e 33$ $=\frac{128}{11}+12\times 3.4965$

$$=11.6364+41.9580=53.5944$$
$$\approx 54 \text{ rabbits/week}$$

2 marks

j. Size of the population at time t is given by
$P(t) = 674 + 12(t-16)\log_e(2t-31)$

Suppose that population takes t weeks to get back to its original number.
Initial size of the population is 1500.
$\therefore$ when $P(t) = 1500$ we have
$1500 = 674 + 12(t-16)\log_e(2t-31)$
$\Rightarrow 413 = 6(t-16)\log_e(2t-31)$

$\therefore t$ is a solution of the above equation.

1 mark

Question 3

a. **i.**
This shape can be considered to be a sphere plus a cylinder. The volume is
$V = V_{sphere} + V_{cylinder}$
$= \frac{4\pi r^3}{3} + \pi r^2 h = \frac{\pi r^2}{3}(4r + 3h)$

2 marks

ii Surface area of this capsule can be considered to be surface area of a sphere plus surface area of the curved surface of a cylinder.
$S = 4\pi r^2 + 2\pi rh$
$= 2\pi r(2r + h)$

2 marks

b. **i.** Given that $V = \pi a^3\ cm$

$\therefore\ \pi a^3 = \frac{\pi r^2}{3}(4r + 3h)$

$\Rightarrow\ h = \frac{a^3}{r^2} - \frac{4r}{3}$

1 mark

ii. To find the domain for h consider the above expression. we know that
$h \geq 0\ \Rightarrow\ \frac{a^3}{r^2} - \frac{4r}{3} \geq 0$

$\Rightarrow \frac{3a^3}{4} \geq r^3$

$\therefore \left(\frac{3}{4}\right)^{\frac{1}{3}} a \geq r > 0$

2 marks

iii

$$S = 2\pi r\left(2r + \frac{a^3}{r^2} - \frac{4r}{3}\right)$$

$$= 2\pi r\left(\frac{a^3}{r^2} + \frac{2r}{3}\right) = 2\pi r^2\left(\frac{a^3}{r^3} + \frac{2}{3}\right)$$

1 mark

c. **i.**

$$\frac{dS}{dr} = -\frac{2\pi a^3}{r^2} + \frac{8\pi r}{3}$$

Turning points are given by $\frac{dS}{dr} = 0$

$$-\frac{2\pi a^3}{r^2} + \frac{8\pi r}{3} = 0$$

$$\therefore \Rightarrow r^3 = \frac{3a^3}{4}$$

$$\Rightarrow r = \left(\frac{3}{4}\right)^{\frac{1}{3}} a$$

2 marks

ii.

when $r = \frac{a}{2}$, $\frac{dS}{dr} = -4\pi a + \frac{4\pi a}{3} < 0$

$\therefore \left(\frac{dS}{dr}\right)_{\left(r < \left(\frac{3}{4}\right)^{\frac{1}{3}} a\right)} < 0$ and

when $r = a$, $\frac{dS}{dr} = -2\pi a + \frac{8\pi a}{3} > 0$

$\therefore \left(\frac{dS}{dr}\right)_{\left(r > \left(\frac{3}{4}\right)^{\frac{1}{3}} a\right)} > 0$

2 marks

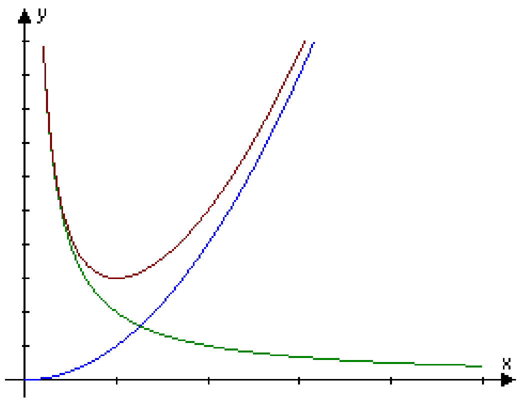
Hence $r = \left(\frac{3}{4}\right)^{\frac{1}{3}} a$ is a minimum point

Corresponding value of S is given by

$$S_{\min} = 4\pi a^2 \left(\frac{3}{4}\right)^{\frac{2}{3}}$$

iii. $S = \dfrac{2\pi a^3}{r} + \dfrac{4\pi r^2}{3}$

$$y = \frac{2\pi a^3}{r} + \frac{4\pi r^2}{3}$$



3 marks

Question 4

a. Let p the probability that he gets a particular answer correct. Then $p = \frac{1}{5}$.

1 mark

b. Binomial distribution with $n = 25$ and $p = \frac{1}{5}$. **1 mark**

c. Random variable X denote the number of correct answers. Then the distribution of **X**

is $P(X = x) = \binom{25}{x}\left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}$ where

$x = 0,1,2...25$

1 mark

d. The mean is given by

$$E(X) = np$$
$$= 25 \times \frac{1}{5} = 5$$

1 mark

Variance of X is given by
$Var(X) = npq$

$$= 25 \times \frac{1}{5} \times \frac{4}{5}$$

$$= 4$$

1 mark

The standard deviation is given by

$$\sigma = \sqrt{Var(X)}$$

$$\therefore \quad = \sqrt{4} = 2$$

1 mark

e.

$$P(X = 0) = \binom{25}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{25}$$

$$= 0.0038$$

1 mark

$P(X > M) = 0.01$ 1 mark **1 mark**

$$P\left(\frac{X-\mu}{\sigma} > \frac{M-\mu}{\sigma}\right) = 0.01$$

$P\left(Z > \frac{M-5}{2}\right) = 0.01$ 1 mark **1 mark**

$$\Rightarrow \quad \frac{M-5}{2} = 2.33$$

$\therefore \quad M = 9.66 \approx 10$ 1 mark **1 mark**