VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2004

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures									Letter:
Words									

MATHEMATICAL METHODS (CAS) PILOT STUDY

Written examination 2

(Analysis task)

Monday 8 November 2004

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.45 am (1 hour 30 minutes)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
4	4	55

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and one approved CAS calculator (memory may be retained) and/or one scientific calculator. For the TI-92, Voyage 200 or approved computer based CAS, their full functionality may be used, but other programs or files are not permitted.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 13 pages, with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions

- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Consider the function $f: R \to R, f(x) = (x-1)^2(x-2) + 1$.

a. Find $f'(x)$

1 mark

b. The coordinates of the turning points of the graph of $y = f(x)$ are $(a, 1)$ and $\left(b, \frac{23}{27}\right)$.
Find the values of a and b.

2 marks

c. Find the real values of p for which the equation $f(x) = p$ has exactly one solution.

2 marks

Question 1 – continued

d. For the following, k is a positive real number.

i. Describe a sequence of transformations which maps the graph of $y = f(x)$ onto the graph of $y = f\left(\frac{x}{k}\right) - 1$.

ii. Find the x-axis intercepts of the graph of $y = f\left(\frac{x}{k}\right) - 1$ in terms of k.

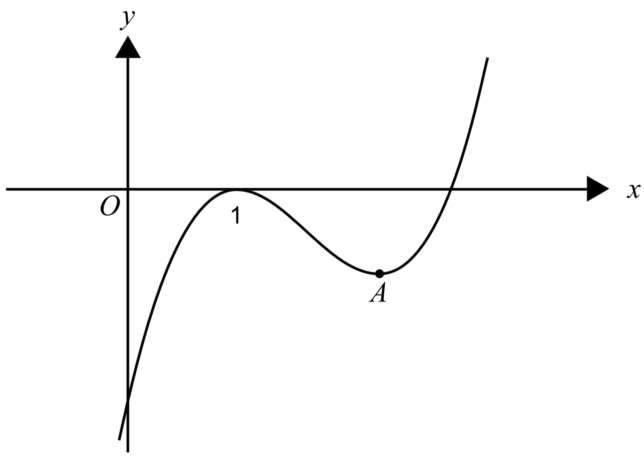
iii. Find the area of the region bounded by the graph of $y = f\left(\frac{x}{k}\right) - 1$ and the x-axis in terms of k.

2 + 2 + 2 = 6 marks

e. Find the real values of h for which **only one** of the solutions of the equation $f(x + h) = 1$ is positive.

2 marks

f. The graph of $y = (x - 1)^2(x - a)$ where $a > 1$ is shown below.

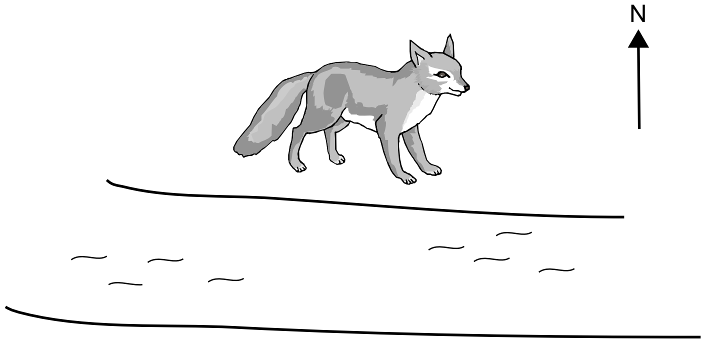


Find the exact value of a such that the local minimum at point A lies on the line with equation $y = -4x$.

3 marks

Total 16 marks

Question 2



A fox hunts each night in one of two areas, either on the north side of a creek or on the south side. The side it hunts on each night depends only on the side it hunted on the night before. If the fox hunts on the north side of the creek one night, then the probability of the fox hunting on the north side of the creek the next night is $\frac{2}{5}$.

The transition matrix for the probabilities of the fox hunting on either side of the creek given the side of the creek hunted on the previous night is $\begin{bmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$.

a. If the fox hunts on the south side of the creek one night, what is the exact probability that it hunts on the north side of the creek the next night?

1 mark

b. Suppose the fox hunts on the north side of the creek on Monday night. What is the exact probability that it will hunt on the north side of the creek on Thursday night of the same week?

3 marks

c. In the long term, what percentage of nights, to the nearest per cent, will the fox hunt on the north side of the creek?

3 marks

The time, t, in hours that the fox spends hunting each night is independent of the area it hunts in and is a random variable with probability density function

$$f(t) = \begin{cases} \frac{3}{32}t(4-t) & \text{if} \quad 0 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

d. What is the exact probability that the fox spends longer than 3 hours hunting on a night?

2 marks

e. What is the probability, correct to three decimal places, that the fox spends longer than 3 hours hunting on at least two out of three nights?

2 marks

f. On 10.4% of nights the fox hunts for less than n **minutes**. Find the value of n.

2 marks

Total 13 marks

Question 3

Since the early 1970s, global population growth rates have been declining. The World Bank uses the following model for the population $P(t)$, in thousand millions of a particular group of people (for example, the low income group) at any time after the beginning of 1990.

$$P(t) = P_0 e^{G(t)}$$

where P_0 thousand million is the population of a given group at the beginning of 1990, t is the number of years after the beginning of 1990, and G is a function of t, where

$$G'(t) = a + bt$$

and a and b are real constants.

This model can be applied to various population groups by using appropriate values of P_0, a and b.

a. **i.** This model can be used when $G'(t) \geq 0$.

For a particular group, $a > 0$ and $b < 0$.

Find, **in terms of *a* and *b***, the number of years for which the model can be used for this group.

ii. Show that $G(0) = 0$

iii. Show that $G(t) = at + \frac{1}{2}bt^2$

2 + 2 + 2 = 6 marks

For the low income group, as defined by the World Bank, the population at the beginning of 1990 was three thousand million, with $a = 0.02$ and $b = -0.0002$.

b. **i.** Show that $P(t) = 3e^{(0.02t - 0.0001t^2)}$

ii. What is the greatest number of years for which this model for the low income group can be used?

iii. Using this model, find the rate of change of the population of the low income group with respect to time.

iv. At the beginning of the year 2010, what will be the rate of change of the population (in thousand millions per year) of the low income group? Give your answer correct to three decimal places.

1 + 1 + 1 + 1 = 4 marks

c. For another group, $P = 3e^{0.01t}$, $t \geq 0$, where P thousand million is the population t years after the beginning of 1990.

i. Find a rule for t in terms of P.

ii. Find the year during which the population of this group will be double what its population was at the beginning of the year 1990.

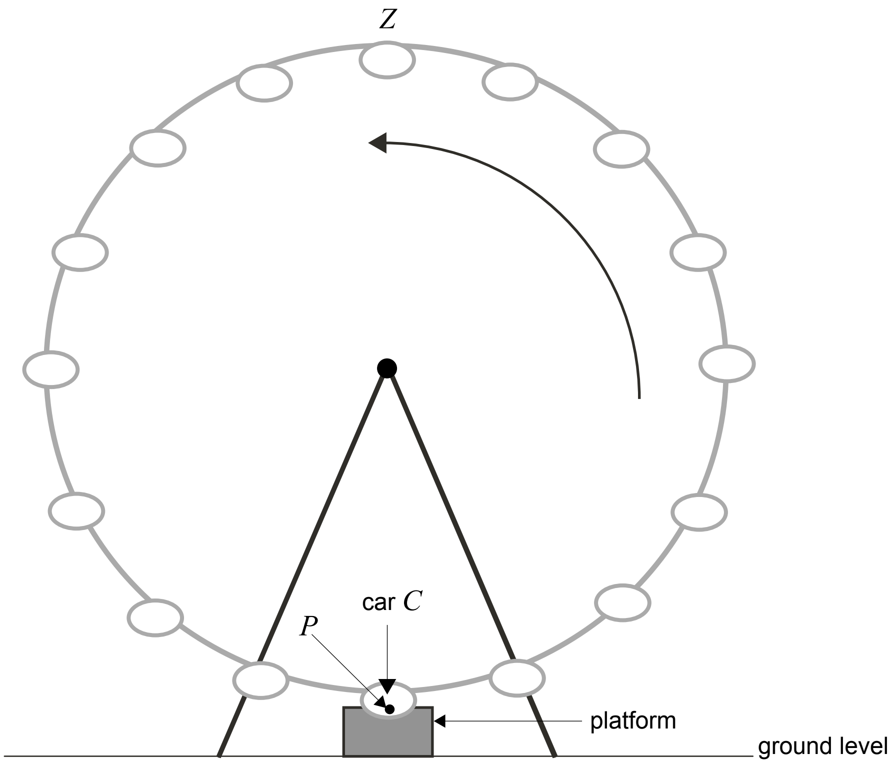
2 + 1 = 3 marks

Total 13 marks

Question 4

A Ferris wheel at a theme park rotates in an anticlockwise direction at a constant rate. People enter the cars of the Ferris wheel from a platform which is above ground level. The Ferris wheel does not stop at any time.

The Ferris wheel has 16 cars, spaced evenly around the circular structure.



A spider attached itself to the point P on the side of car C when the point P was at its lowest point at 1.00 pm.

The height, h metres, of the point P above ground level, at time t hours after 1.00 pm is given by

$$h(t) = 62 + 60 \sin\left(\frac{(5t-1)\pi}{2}\right)$$

a. Write down the maximum height, in metres, of the point P above ground level.

1 mark

b. Write down the minimum height, in metres, of the point P above ground level.

1 mark

c. At what time, after 1.00 pm, does point P first return to its lowest point?

1 mark

d. **i.** Find the time, after 1.00 pm, when the point P first reaches a height of 92 metres above ground level.

ii. Find the **number of minutes** during one rotation when the point P is at least 92 metres above ground level.

2 + 1 = 3 marks

e. **i.** Write down an expression, in terms of t, for the rate of change of h with respect to time.

ii. At what rate (in m/h) is h changing when $t = 1$?

1 + 1 = 2 marks

When point P first reaches position Z, that is, its highest point (see diagram page 11), the spider becomes frightened. It drops down from the car on a thread (which remains vertical at all times) at a rate of 5 metres per minute until it reaches the ground.

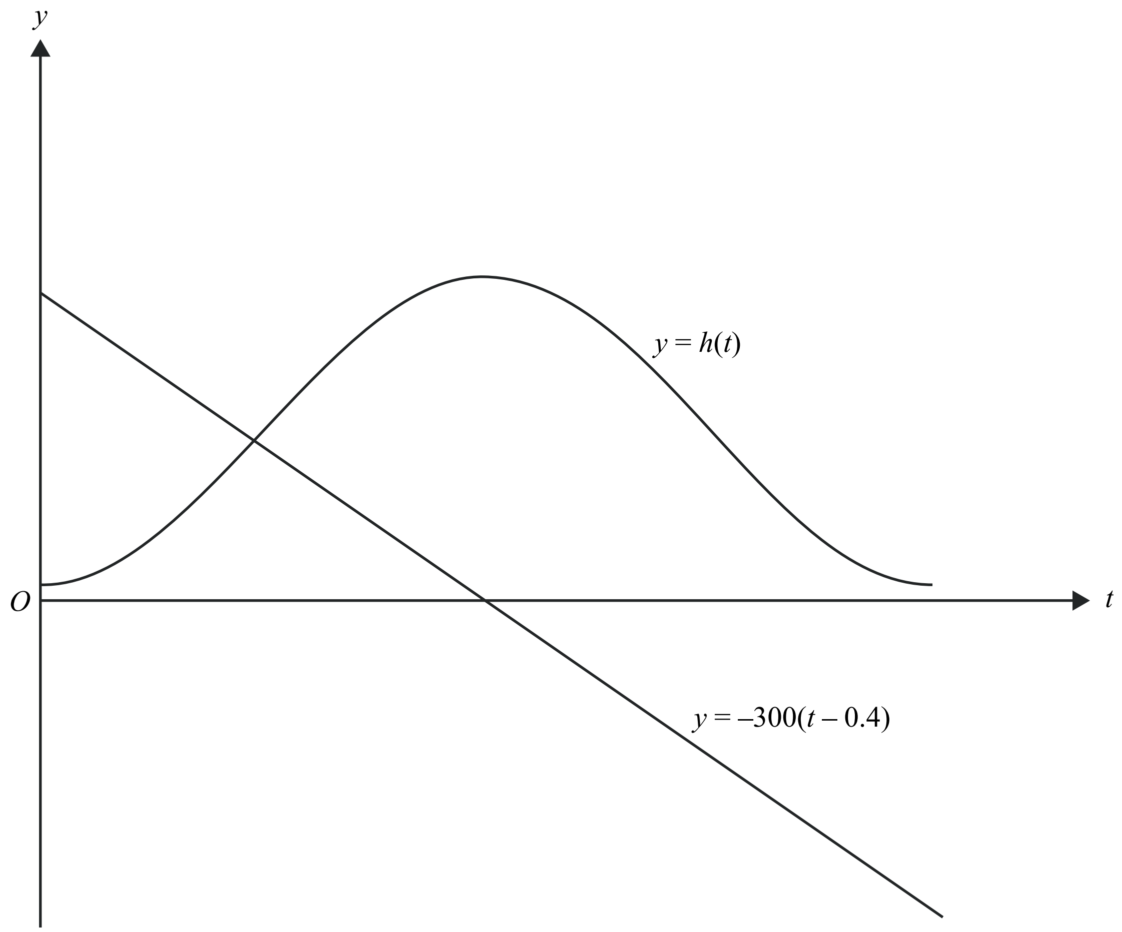
As it drops, the spider's height $s(t)$ metres above ground level at time t (where t is the time in hours after 1.00 pm) is given by

$$s(t) = h(t) - 300(t - 0.4)$$

f. The graph of $y = h(t)$ for the first revolution of the Ferris wheel after 1.00 pm and the graph of $y = -300(t - 0.4)$ are shown together on the axes below.

i. On the diagram label the local maximum point of the graph of $y = h(t)$ with its coordinates.

ii. On the diagram draw a graph which shows the height of the spider above ground level at time t.



iii. Find, to the nearest minute, the time taken from when the spider leaves car C to when it reaches the ground.

1 + 2 + 2 = 5 marks

Total 13 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS (CAS)
PILOT STUDY

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

Mathematical Methods CAS Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e |x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$

$\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$

$\int \cos(ax) dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

average value: $\frac{1}{b-a}\int_a^b f(x)dx$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

Discrete distributions			
	$\Pr(X = x)$	mean	variance
general	$p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2 p(x)$ $= \sum x^2 p(x) - \mu^2$
binomial	${}^nC_x p^x (1-p)^{n-x}$	np	$np(1-p)$
hypergeometric	$\frac{{}^DC_x\ {}^{N-D}C_{n-x}}{{}^NC_n}$	$n\frac{D}{N}$	$n\frac{D}{N}\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right)$

Continuous distributions			
	$\Pr(a < X < b)$	mean	variance
general	$\int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$ $= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$
normal	If X is distributed $N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, then Z is distributed $N(0, 1)$, $f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$		

Table 1 Normal distribution – cdf

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	4	8	12	16	20	24	28	32	36
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	4	8	12	16	20	24	28	32	35
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	4	8	12	15	19	23	27	31	35
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	4	8	11	15	19	23	26	30	34
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	4	7	11	14	18	22	25	29	32
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	3	7	10	14	17	21	24	27	31
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	3	6	10	13	16	19	23	26	29
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7793	.7823	.7852	3	6	9	12	15	18	21	24	27
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	3	6	8	11	14	17	19	22	25
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	3	5	8	10	13	15	18	20	23
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	2	5	7	9	12	14	16	18	21
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830	2	4	6	8	10	12	14	16	19
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	2	4	6	7	9	11	13	15	16
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	2	3	5	6	8	10	11	13	14
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1	3	4	6	7	8	10	11	13
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1	2	4	5	6	7	8	10	11
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1	2	3	4	5	6	7	8	9
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1	2	3	3	4	5	6	7	8
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1	1	2	3	4	4	5	6	6
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1	1	2	2	3	4	4	5	5
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	0	1	1	2	2	3	3	4	4
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	0	1	1	2	2	2	3	3	4
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	0	1	1	1	2	2	2	3	3
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	0	1	1	1	1	2	2	2	2
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	0	0	1	1	1	1	1	2	2
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	0	0	0	1	1	1	1	1	1
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	0	0	0	0	1	1	1	1	1
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	0	0	0	0	0	1	1	1	1
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	0	0	0	0	0	0	0	1	1
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	0	0	0	0	0	0	0	0	0
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	0	0	0	0	0	0	0	0	0
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	0	0	0	0	0	0	0	0	0
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	0	0	0	0	0	0	0	0	0
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	0	0	0	0	0	0	0	0	0
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	0	0	0	0	0	0	0	0	0
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	0	0	0	0	0	0	0	0	0
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0

END OF FORMULA SHEET