2003 Mathematical Methods (CAS) Pilot Study GA 3: Written examination 2

GENERAL COMMENTS

The number of students presenting for Mathematical Methods (CAS) Examination 2 in 2003 was 268 with the range of marks from 0 to 55. Student responses showed that the paper was accessible and provided sufficient opportunities to demonstrate what they knew. There were excellent papers presented by several students, two students achieved a perfect score and 20% of students scored over 80% of the available marks. The median and mean marks for the paper were 34.5 and 32 respectively, with, 67% of the students scoring over half of the available marks and 78% of the students scoring over 40% of the available marks. Twenty one students scored less than 20% of the available marks.

Generally the symbolic manipulation facility of CAS was used well. There was no discernable advantage seen by the assessors of one computer algebra system compared to another. The generally high level of proficiency with the systems was demonstrated in particular in Question 3a where 92% of students achieved full marks, Question 4a where 71% of the students achieved full marks and Question 4b where 79% of the students achieved full marks. As in 2002, sometimes students did not give answers in exact form although this was explicitly asked for. An important aspect of CAS use is their ability to handle exact values readily. For example in Question 3ci students were required to find the equation of the tangent to the graph of $y = f(x)$. The equation of the tangent is $y = e^{-2}x$. Students often used the numerical facility of their CAS and obtained approximate results such as $y = 0.135x$ or, more surprisingly, $y = 0.135x + 10^{-23}$

Graph sketching is a skill that would benefit from further attention, with only 15% of students achieving full marks on Question 4dii. In some cases this could likely be attributed to poor use of the graphing facility of the CAS, where the window was not well chosen. It was anticipated by examiners that students would sketch the graph of the inverse by reflecting the graph of the original function in the line with equation $y = x$. The graph of the function could be obtained through correct use of the graphing facility.

Question 1, the probability question, can be done quickly using CAS without any working being shown. If the question is worth more than 1 mark, and no working is shown, students will not gain full marks. The instructions at the beginning of the paper state that if more than one mark is available for a question appropriate working must be shown. Students should carefully note this when giving their responses, and respond accordingly.

For example, in Question 1b, a correct answer (ie. 0.17) with some suitable working, gained 2 marks. An answer of 0.16 with no working scored 0 marks. One mark out of the two was awarded if the answer was incorrect but working, such as one or more of the following, was shown:

- a normal distribution diagram with appropriate mean and value marked, and appropriate area indicated

or

- correct interpretation with standard normal distribution; and recognition of normal and correct probability statement with *d*.

Many students lost marks because they:

- did not answer the question asked
- gave decimal answers when an exact answer was required
- gave the wrong number of decimal places
- did not pay sufficient attention to detail in sketching graphs.

When students present working and develop solutions, they should ensure that they use conventional mathematical expressions, symbols, notation and terminology, rather than calculator syntax.

SPECIFIC INFORMATION

Question 1

1a.

Marks	0	1	2	Average
%	16	15	69	**1.52**

Cumulative normal with $\mu = 140$, $\sigma = 1.2$

$\Pr(X > 141.5) = 0.106$ where X is the length of the rod.

1b.

Marks	0	1	2	Average
%	65	11	24	**0.58**

Pr (too large) = $\frac{0.15}{2} = 0.075$

inverse normal $d = 1.7$

1c.

Marks	0	1	2	Average
%	24	14	62	1.38

Binomial distribution with $n = 12$ and $p = 0.15$

$\Pr(Y = 2) = 0.292$ where Y is the number of rods with a size fault

1d.

Marks	0	1	2	Average
%	16	29	55	1.38

Hypergeometric distribution

$1 – (\Pr (W = 0) + \Pr (W = 1)) = 0.672$ where W is the number of rods in the sample with a size fault

1ei.

Marks	0	1	Average
%	9	91	0.91

$1 – (0.15 + 0.17) = 0.68$

1eii.

Marks	0	1	Average
%	34	66	0.66

$0.17(x – 8) + 0.68(x –5) = 0.85x – 4.76$

1eiii.

Marks	0	1	Average
%	50	50	0.50

$0.85x – 4.76 = 0$ gives $x = 5.60$

1eiv.

Marks	0	1	Average
%	85	15	0.15

Pr (good rod | given rod ready to be sold) = $\frac{0.68}{0.68 + 0.17} = 0.8$

In Question 1b the observation that $\Pr (140 – d \le X \le 140 + d) = 0.85$ or $\Pr (X \ge 140 + d) = 0.075$ was rewarded with 1 mark. For some students this problem was confused with complement probability. Most students did not recognise that the solution process involved the use of the inverse normal function. Questions 1a, b and c were well done with over half of the students achieving full marks. A common error in Question ei was the failure to not give the answer to the nearest cent. Question 1e iv was not recognised as conditional probability with a large number of students giving the proportion of good rods as 68% which is the percentage of all rods which are good rather than the result for the restricted sample space.

Question 2

2a.

Marks	0	1	Average
%	8	92	0.92

2.83m

2b.

Marks	0	1	2	Average
%	26	36	38	1.12

gradient = $\frac{dy}{dx} = \sin\left(\frac{x}{2}\right)$

Note that $\sin\left(\frac{x}{2}\right) \le 1$ for all x for the range of the sin function.

2ci.

Marks	0	1	2	Average
%	6	7	87	1.81

Area = $\int_{-4}^{4} \left(2 – 2\cos\left(\frac{x}{2}\right)\right) dx$

2cii.

Marks	0	1	Average
%	15	85	0.85

Answer: 8.73

2di.

Marks	0	1	2	Average
%	16	13	71	**1.54**

$1 = 2 - 2\cos\left(\frac{x}{2}\right) \Rightarrow x = \frac{2\pi}{3}$

2dii.

Marks	0	1	2	Average
%	34	15	51	**1.17**

gradient of tangent at $\frac{2\pi}{3}$

$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

gradient of normal $= -\frac{2}{\sqrt{3}}$

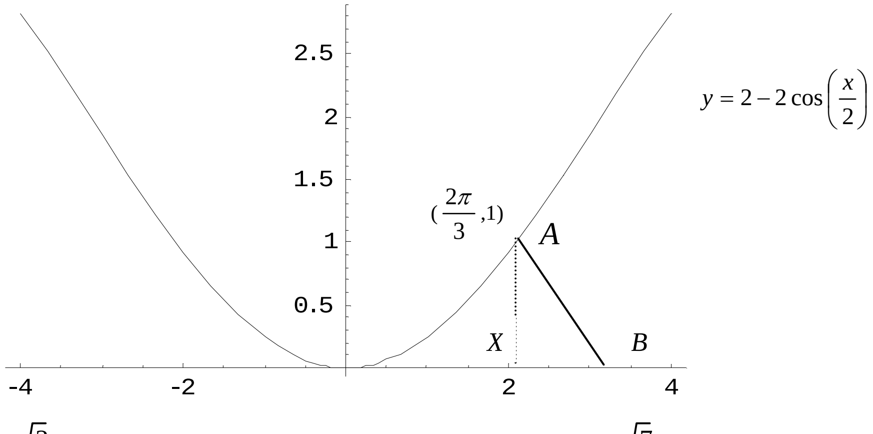
2diii.

Marks	0	1	2	3	Average
%	32	30	14	24	**1.29**

x-coordinate of B is $\frac{2\pi}{3} + \frac{\sqrt{3}}{2}$

AB $= \sqrt{\frac{3}{4}+1} = \frac{\sqrt{7}}{2}$ (see below)

Question 2a was very well done, with the answer generally given to the required accuracy. No students used the incorrect circular function mode. In order to obtain full marks on Question 2b the student was required to find the derivative of the function and then observe that $\sin\left(\frac{x}{2}\right) \leq 1$ for all *x*. Of the cohort, 74% students realised that it was necessary to find the derivative but approximately only half of these students went on to achieve full marks for the question. Questions 2ci and 2cii were on the whole very well done. Several students wrote $\int_{-4}^{4}(f(x))dx$ without defining *f*. This answer resulted in half of the available marks being awarded. Question di was well answered and a good number of students obtained the gradient of the normal successfully in Question dii. Question diii could be answered efficiently by considering a suitable right angle triangle as shown below.



$\frac{AX}{XB} = \frac{2}{\sqrt{3}}$ and $XB = \frac{\sqrt{3}}{2}$ since AX = 1. Pythagoras' Theorem now gives $AB = \frac{\sqrt{7}}{2}$

Question 3

3a.

Marks	0	1	Average
%	8	92	**0.92**

$f'(x) = 3x^2e^{-2x} - 2x^3e^{-2x}$

3b.

Marks	0	1	2	Average

%	11	30	59	**1.48**

$f'(x) = 0$ when $(3x^2 - 2x^3) = 0$ since $e^{-2x} > 0$

so $x = 0$ or $x = \frac{3}{2}$

Stationary point of inflexion at (0,0); local maximum at $(1.5, \frac{27e^{-3}}{8})$

3ci.

Marks	**0**	**1**	**2**	**3**	**Average**
%	11	15	7	67	**2.31**

$f(1) = e^{-2}$ and $f'(1) = e^{-2}$

$y - e^{-2} = e^{-2}(x - 1)$. Therefore $y = e^{-2}x$

3cii.

Marks	**0**	**1**	**Average**
%	28	72	**0.72**

$f'(0) = 0$ so equation is $y = 0$.

3ciii.

Marks	**0**	**1**	**2**	**3**	**Average**
%	82	12	1	5	**0.29**

Line through O is $y = mx$, so $m = \frac{y}{x}$ = gradient of curve.

Then $f'(x) = x^2e^{-2x}(3 - 2x) = \frac{y}{x} = x^2e^{-2x}$

$\Rightarrow x^2e^{-2x}(3 - 2x) - x^2e^{-2x} = 0$

$\Rightarrow x^2e^{-2x}(2 - 2x) = 0 \Rightarrow x = 0$ or 1

3di.

Marks	**0**	**1**	**2**	**Average**
%	38	10	52	**1.13**

$$\int_0^{\infty}(kx^3e^{-2x})dx = \frac{3k}{8}$$

so $\frac{3k}{8} = 1$ and $k = \frac{8}{3}$

3dii.

Marks	**0**	**1**	**2**	**Average**
%	42	20	38	**0.95**

$$\int_0^{a}(g(x))dx = 0.5$$

$a = 1.84$

Question 3a was done well. In Question 3b students often did not obtain all the available marks as they did not answer the question in full such as y coordinates were not given.

In Question 3ci and 3cii students were required to give an equation but many wrote answers as just $e^{-2}x$, or Tangent = $e^{-2}x$, both of which were unacceptable. Partial marks were obtained for obtaining the gradient.

Question 3ciii required a reasoned argument to be provided. In general this question was not done well, and many students did not produce a coherent argument. Marks were given if a student attempted to find a general equation to a tangent or noted that the gradient of the line segment from a point on the curve to the origin must equal the gradient of the curve at that point if the tangent was to pass through the origin. Questions 3di and 3dii involved sensible use of CAS numerical and/or graphical functionality. The solution of 3dii can also quickly be obtained by systematic trial and error, checking the definite integral, for values of the upper terminal.

Question 4

4a.

Marks	**0**	**1**	**2**	**Average**
%	20	9	71	**1.50**

$\frac{dx}{dt} = \frac{3(5 + t^2) - 6t^2}{(5 + t^2)^2} = 0$ for maximum $\Rightarrow 15 - 3t^2 = 0$

$\Rightarrow t = \pm\sqrt{5}$. But $t \geq 0$, so $t = \sqrt{5}$

When $t = \sqrt{5}$, $x = \dfrac{3\sqrt{5}}{10}$

4b.

Marks	0	1	Average
%	21	79	0.79

When $t = 0$, $\dfrac{dx}{dt} = \dfrac{1}{3}$

The rate of absorption is $\dfrac{1}{3}$ mg/L/hour

4c.

Marks	0	1	2	3	Average
%	14	11	22	53	2.14

$$\frac{3t}{5+t^2} = 0.4 \Rightarrow 0.4t^2 - 3t + 2 = 0$$

$$\Rightarrow t = \frac{15 \pm \sqrt{145}}{4}.$$

Time required is the difference of roots $= \dfrac{\sqrt{145}}{2}$, the tranquilliser is effective for $\dfrac{\sqrt{145}}{2}$ hours

4di.

Marks	0	1	Average
%	51	49	0.49

Least value $= \sqrt{5}$

4dii.

Marks	0	1	2	3	Average
%	46	20	19	15	1.01

A graph with vertical asymptote $t = 0$. The endpoint has coordinates $\left(\dfrac{3\sqrt{5}}{10}, \sqrt{5}\right)$

4diii.

Marks	0	1	2	3	Average
%	21	19	24	36	1.74

$t = \dfrac{3x}{5+x^2}$

$\Rightarrow tx^2 - 3x + 5t = 0$

$\Rightarrow x = \dfrac{3 \pm \sqrt{(9-20t^2)}}{2t}$

But we want $x \geq \sqrt{5}$,

so $x = \dfrac{3 + \sqrt{(9-20t^2)}}{2t}$

4e.

Marks	0	1	2	3	Average
%	66	14	6	14	0.67

$\dfrac{dx}{dt} = \dfrac{3(p+t^2) - 6t^2}{(p+t^2)^2} = 0$ for maximum $\Rightarrow 3p - 3t^2 = 0$

$\Rightarrow t = \pm\sqrt{p}$. But $t \geq 0$, so $t = \sqrt{p}$

Therefore maximum $= \dfrac{3\sqrt{p}}{p+p} = \dfrac{3}{2\sqrt{p}}$

for $\dfrac{3}{2\sqrt{p}} < 1$ which implies $p > \dfrac{9}{4}$

Generally Questions 4a and b were done well. A small number of students did not give exact values for the coordinates. Question 4c was also successfully completed by many students. This question was certainly made accessible by sensible use of CAS and many students demonstrated not only their understanding of the concept involved but their capacity to use CAS efficiently. The graphing question of 4dii was not done as well. Students used the CAS in diii to successfully solve the quadratic equation, however the failure to recognise that only one of the solutions gave the inverse caused many students to lose a mark. In Question 4e the least value of p can only be described to the extent of giving a greatest lower bound of $\dfrac{9}{4}$, that is, $p > \dfrac{9}{4}$. Full marks were awarded for noting the greatest lower bound value together with an argument based on the fact that the maximum value of $y(t)$ occurs when $t = \sqrt{p}$. The least value could be determined to an accuracy given by the student. A graphical argument was not sufficient to obtain full marks.

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