Trial Examination 2002

VCE Mathematical Methods Units 3 & 4

Examination 1: Facts, Skills and Applications Task

Suggested Solutions

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PART I

Question 1

This graph has the basic shape of a $y = -\sin x$ graph. $\frac{3}{4}$ of its period is $\frac{3\pi}{4}$, so its period is π.

Period $= \frac{2\pi}{n} \Rightarrow n = 2$. The amplitude is a and it has been translated down by a units. The equation is therefore $y = -a\sin 2x - a$.

Answer E

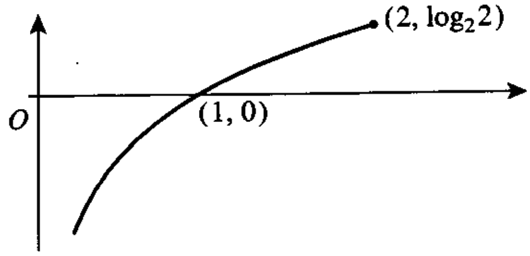
Question 2

The term containing x^2 is ${}^6C_2(3x)^2(-2)^4 = 15(3)^2(-2)^4x^2 = 2160x^2$.

Answer C

Question 3

The graph of $f(x)$ is shown below.



The range of f is $(-\infty, 1]$, as $\log_2 2 = 1$.

Answer D

Question 4

$$\begin{aligned}\log_e x - 3\log_e 2x + 2\log_e 3x &= \log_e x - \log_e(2x)^3 + \log_e(3x)^2 \\ &= \log_e \frac{x \times (3x)^2}{(2x)^3} \\ &= \log_e \frac{9}{8} \\ &= \log_e 9 - \log_e 8.\end{aligned}$$

Answer B

Question 5

$$\begin{aligned}\log_2 x(x-1) &= 1 \\ \therefore x^2 - x &= 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2 \text{ or } -1.\end{aligned}$$

Answer E

Question 6

The graph shown is a negative quartic with x-intercepts a and b. Therefore linear factors of $x - a$ and $x - b$ exist. We notice a turning point on the x-axis when $x = 0$; hence x^2 is a factor in the equation.

We look for $y = -x^2(x - a)(x - b)$, which can also be expressed as $y = x^2(a - x)(x - b)$.

Answer D

Question 7

Use of the graphic calculator is required. Entering `Y1=2sin(3x)` and `Y2=log(3x)`, we observe 3 intersections. Hence there are 3 solutions.

Answer C

Question 8

The graph required has $y = f(x)$ translated b units to the right and dilated by a factor a away from the x-axis in the y direction.

Answer B

Question 9

The easiest way to generate the resultant graph is to add $-g(x)$ to $f(x)$. This results in a y-intercept of $(0, 1)$, and $y \to -\infty$ as $x \to -\infty$ and as $x \to \infty$.

Answer D

Question 10

We look for a reflection in the line $y = x$ and all inverse coordinates will have x and y values interchanged.

Answer A

Question 11

The vertical asymptote $x = a$ results from equating the denominator to zero. This is a negative rectangular hyperbola, hence the denominator is $a - x$. The horizontal asymptote is obtained when b is added to $\frac{c}{a - x}$.

The y-intercept (when $x = 0$) confirms that D is correct.

Answer D

Question 12

$f(x) = e^{x^2 + 1}$. By the chain rule, $f'(x) = e^{x^2 + 1}(2x) = 2xe^{x^2 + 1}$.

Answer A

Question 13

$f(x)$ is a positive cubic, hence $f'(x)$ is a positive quadratic. The stationary points of $f(x)$ are the x-intercepts of $f'(x)$. The turning points of $f'(x)$ correspond to the maximum negative gradient of $f(x)$.

Answer C

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Question 14

$f(x) = \log_e\left(\frac{1}{\sin x}\right) = \log_e((\sin x)^{-1}) = -\log_e(\sin x)$.

Using the chain rule, $f'(x) = -\frac{\cos x}{\sin x} = -\tan x$.

Answer A

Question 15

Using the quotient rule, $\frac{dy}{dx} = \frac{e^x \pi \cos \pi x - (\sin \pi x)e^x}{e^{2x}} = \frac{e^x(\pi \cos \pi x - \sin \pi x)}{e^{2x}}$.

At $x = 0$, $\frac{dy}{dx} = \frac{e^0(\pi \cos 0 - \sin 0)}{e^0} = \pi$.

Alternatively, the product rule could be used with $y = e^{-x}(\sin \pi x)$:

$\frac{dy}{dx} = e^{-x}(\pi \cos \pi x) + (\sin \pi x)(-e^{-x}) = e^{-x}(\pi \cos \pi x - \sin \pi x)$.

At $x = 0$, $\frac{dy}{dx} = e^0(\pi \cos 0 - \sin 0) = \pi$.

Answer C

Question 16

Using the product rule, $\frac{dy}{dx} = e^x\left(\frac{1}{x}\right) + (\log_e 2x)e^x = e^x\left(\frac{1}{x} + \log_e 2x\right)$.

Answer A

Question 17

At $x = 0$, $y = e^{-\frac{1}{2}(0)} - 1 = 0$.

The gradient of the tangent, m_T is given by $\frac{dy}{dx} = -\frac{1}{2}e^{-\frac{x}{2}}$. At $x = 0$, $m_T = -\frac{1}{2}$.

$m_T m_N = -1$, so at $x = 0$, $m_N = \frac{-1}{-\frac{1}{2}} = 2$.

Hence the equation of the normal at $(0, 0)$ is $y - 0 = 2(x - 0)$

$$y = 2x.$$

Answer C

Question 18

Using the left rectangle approximation for

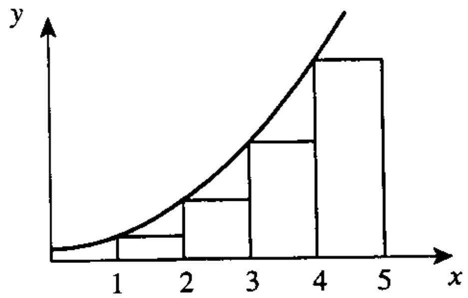
$y = x^2 + 1$

When $x = 1$, $y = 2$.

When $x = 2$, $y = 5$.

When $x = 3$, $y = 10$.

When $x = 4$, $y = 17$.



Approximate area $= 2 + 5 + 10 + 17$

$= 34$ square units.

Answer B

Question 19

$$\int \frac{1}{2(5x+2)^2}dx = \frac{1}{2}\int (5x+2)^{-2}dx$$

$$= \frac{1}{2}\left[\frac{(5x+2)^{-1}}{5(-1)}\right] + C$$

$$= -\frac{1}{10}\left(\frac{1}{5x+2}\right) + C.$$

An antiderivative is $-\frac{1}{10}\left(\frac{1}{5x+2}\right)$.

Answer E

Question 20

$$\text{Area} = \int_{-1}^{2} -x^2 - (-x-2)dx$$

$$= \int_{-1}^{2} -x^2 + x + 2\ dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right]_{-1}^{2}$$

$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= \frac{9}{2}\text{square units.}$$

Answer B

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Question 21

$$\int_a^{\frac{3\pi}{4}} \sin 2x = 0$$

$$\left[-\frac{1}{2}\cos 2x\right]_a^{\frac{3\pi}{4}} = 0$$

$$-\frac{1}{2}\cos\frac{3\pi}{2} - \left(-\frac{1}{2}\cos 2a\right) = 0$$

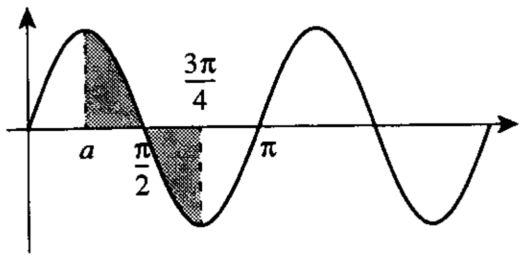
$$0 + \frac{1}{2}\cos 2a = 0$$

$$\cos 2a = 0$$

$$2a = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$a = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$$

OR



Balancing areas above and below the x-axis, $a = \frac{\pi}{4}$.

Answer A

Question 22

$\mathrm{E}(A) = np = 2$ and $\mathrm{Var}(A) = \frac{3}{2} = np(1-p)$.

$$\therefore \frac{3}{2} = 2(1-p)$$

$$\therefore (1-p) = \frac{3}{4}$$

$$\therefore p = \frac{1}{4}$$

As $\mathrm{E}(A) = np = 2$, $n = \frac{2}{\frac{1}{4}} = 8$.

For a binomial random variable, $\Pr(A = 2) = {}^nC_2 p^2(1-p)^{n-2} = {}^8C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6$.

Answer D

Question 23

This is a hypergeometric experiment. Let X be the number who favour all-year protection.

Then $\Pr(X = 3) = \dfrac{\binom{5}{3}\binom{3}{1}}{\binom{8}{4}} = \dfrac{10 \times 3}{70} = \dfrac{3}{7}$.

Answer C

Question 24

Z has a standard normal distribution.

We require area A.

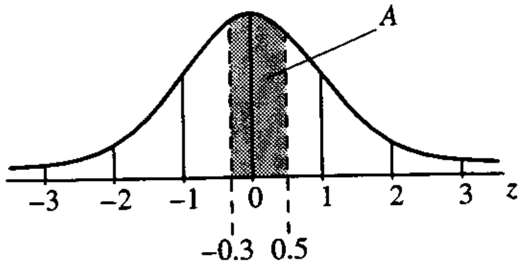
$A = \Pr(Z < 0.5) - \Pr(Z < -0.3)$

$= \Pr(Z < 0.5) - \Pr(Z > 0.3)$

$= \Pr(Z < 0.5) - (1 - \Pr(Z < 0.3))$

$= \Pr(Z < 0.5) - 1 + \Pr(Z < 0.3)$

$= \Pr(Z < 0.5) - 1 + \Pr(Z > -0.3)$.



Answer B

Question 25

Let X be the number of insects zapped in the tray.

x	0	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{7}{20}$	$\frac{4}{20}$	$\frac{1}{20}$

The mean number of insects zapped is given by

$$\mathrm{E}(X) = 0 + \frac{2}{20} + \frac{2}{20} + \frac{12}{20} + \frac{28}{20} + \frac{20}{20} + \frac{6}{20} = \frac{7}{2}.$$

Answer C

Question 26

This is a hypergeometric experiment with $n = 10$, $N = 60$ and $D = 35$.

Let X be the number of yellow tees selected. Then $\mathrm{E}(X) = n\frac{D}{N} = \frac{35}{60} \times 10 = 5.83$

and $\mathrm{Var}(X) = \frac{nD(N-D)(N-n)}{N^2(N-1)} = \frac{(10 \times 35)(25)(50)}{3600(59)} = 2.06$.

Answer D

Question 27

X is hypergeometric, Y is binomial. By experiment it can be found that $\mathrm{E}(X) = \mathrm{E}(Y)$ and $\mathrm{SD}(X) < \mathrm{SD}(Y)$ when n is small. The graph in A is the only representation of this situation.

Answer A

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PART II

Question 1

a. The smallest x value is obtained by letting $2x - 1 = 0$, i.e. when $x = \frac{1}{2}$.

There is no upper limit on the value of x. Hence the maximal domain is $\left(\frac{1}{2}, \infty\right)$. [A]

(Note that $x = \frac{1}{2}$ is not included in the domain as $\log_e 0$ is indeterminate.)

b. For the inverse function, swap x and y,

i.e. $x = \log_e(2y - 1)$ [M]

$e^x = 2y - 1$

$y = \frac{e^x + 1}{2}$

$\therefore f^{-1}(x) = \frac{e^x + 1}{2}$ [A]

c. The domain of $f^{-1}(x)$ is the range of $f(x)$ and the range of $f^{-1}(x)$ is the domain of $f(x)$.

So, the range of $f^{-1}(x) = \left(\frac{1}{2}, \infty\right)$. [A]

Question 2

$\sqrt{3} + 2\sin\frac{x}{2} = 0$

$2\sin\frac{x}{2} = -\sqrt{3}$

$\sin\frac{x}{2} = -\frac{\sqrt{3}}{2}$. [A]

$\therefore \frac{x}{2} = -\frac{\pi}{3}, -\pi + \frac{\pi}{3}$

$\therefore x = -\frac{2\pi}{3}, -\frac{4\pi}{3}$. [A][A]

Question 3

a. $y = x^2 \log_e 3x$.

Using the product rule, [M]

$\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + (\log_e 3x)2x$

$= x + 2x\log_e 3x$. [A]

b. The gradient of the tangent is $\frac{dy}{dx}$. When $x = \frac{1}{3}$, $\frac{dy}{dx} = \frac{1}{3} + 2 \times \frac{1}{3} \times \log_e 1$

$= \frac{1}{3}$, as $\log_e 1 = 0$. [A]

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c. The partial equation of the tangent is given by $y = \frac{1}{3}x + c$.

Substitute in $\left(\frac{1}{3}, 0\right)$: $0 = \frac{1}{9} + c$ [M]

$\therefore c = -\frac{1}{9}$.

Hence the equation of the tangent is $y = \frac{1}{3}x - \frac{1}{9}$. [A]

Question 4

An approximation for δy is $\frac{dy}{dx} \times \delta x = e^x \times h$. [M]

When $x = \log_e 2$, $\delta y \approx e^{\log_e 2} \times h$

$= 2h$. [A]

Question 5

a. Let X be the number of Australians in favour of the Government's decision.

$\therefore \text{E}(X) = np$

$= 1769 \times 0.54$

$= 955.26$

$\therefore \text{E}(X') = 1769 - 955.26$

$= 813.74$.

So we expect 813 (or 814) people to be non-supporters of the Government decision. [A]

(Alternatively, $\text{E}(X') = 1769 \times (1 - 0.54) = 813.74$.)

b. $X \sim \text{Bi}(n = 5, p = 0.54)$ and $\Pr(X \geq 2) = 1 - [\Pr(X = 0) + \Pr(X = 1)]$. [M]

$\Pr(X = 0) = {}^5C_0(0.54)^0(0.46)^5$

$= (0.46)^5$

$= 0.0206$.

$\Pr(X = 1) = {}^5C_1(0.54)^1(0.46)^4$

$= 0.1209$.

Hence $\Pr(X \geq 2) = 1 - [0.0206 + 0.1209]$

$= 0.8585$

$= 86\%$, to the nearest percentage point. [A]

(or $1 - \text{binomcdf}(5, 0.54, 1) = 1 - 0.1415 = 0.8585$)

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Question 6

a. $X \sim N(67, \sigma^2)$.

$$\Pr(X < 50) = 0.1$$

$$\therefore \Pr\left(Z < \frac{50-67}{\sigma}\right) = 0.1$$ [M]

$$\therefore \Pr\left(Z < -\frac{17}{\sigma}\right) = 0.1$$

$$\therefore 1 - \Pr\left(Z < \frac{17}{\sigma}\right) = 0.1$$

$$\therefore \Pr\left(Z < \frac{17}{\sigma}\right) = 0.9$$

$$\frac{17}{\sigma} = 1.28155$$ [A]

Hence $\sigma = 13.27$. [A]

b. Let X be the number of prize disks in a box.

$$\Pr(X = 0) = \binom{6}{0}(0.13)^0(0.87)^6$$

$$= 0.43363$$

$$\Pr(X = 1) = \binom{6}{1}(0.13)^1(0.87)^5$$

$$= 0.38877$$

$$\Pr(X = 2) = \binom{6}{2}(0.13)^2(0.87)^4$$

$$= 0.14523$$ [A]

$$\therefore \Pr(X < 3 \mid X \geq 1) = \frac{\Pr(1 \leq X \leq 2)}{\Pr(X \geq 1)}$$ [M]

$$= \frac{0.38877 + 0.14523}{1 - 0.43363}$$ [A]

$$= 0.9428$$

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