2002 Mathematical Methods
Written examination 1 (facts, skills and applications)
Suggested answers and solutions

Part I (Multiple-choice) Answers

1. **E**	2. **E**	3. **D**	4. **E**	5. **C**
6. **A**	7. **A**	8. **C**	9. **B**	10. **A**
11. **D**	12. **D**	13. **C**	14. **B**	15. **D**
16. **C**	17. **A**	18. **B**	19. **A**	20. **B**
21. **B**	22. **D**	23. **D**	24. **A**	25. **A**
26. **B**	27. **E**			

1. Amplitude = 1 **[E]**

Period = 4π

$\Rightarrow \frac{2\pi}{n} = 4\pi$

$n = \frac{2\pi}{4\pi}$

$n = \frac{1}{2}$

This eliminates A and C.
Shape is a cosine curve translated 1 unit up, so the answer is $y = 1 + \cos\left(\frac{x}{2}\right)$

OR

Can use graphics calculator in RAD mode or substitute $x = 0$ in remaining answers to find when $y = 2$.

2. $\sin(2x) = 1 \quad x \in [0, 4\pi]$ **[E]**

$2x = \sin^{-1}(1) \quad 2x \in [0, 8\pi]$

$2x = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 6\pi, \frac{\pi}{2} + 8\pi$

$2x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\text{sum} = \frac{\pi}{4} + \frac{5\pi}{4} + \frac{9\pi}{4} + \frac{13\pi}{4}$

$= \frac{28\pi}{4}$

$= 7\pi$

3. $y = 18 - 5\sin\left(\frac{\pi t}{12}\right)$ **[D]**

max occurs when:

$\sin\left(\frac{\pi t}{12}\right) = -1$ i.e. $y = 18 - 5(-1)$

$= 18 + 5$

$= 23$ °C

$\frac{\pi t}{12} = \sin(-1)$

$\frac{\pi t}{12} = \frac{3\pi}{2}$

$t = 18$ hours after midnight is 6 pm

OR

Can use graphics calculator in RAD mode.

Type $y_1 = 18 - 15\sin\left(\frac{\pi x}{12}\right)$, $X_{min} = 0$, $X_{max} = 24$, ZOOMFIT

Find max turning point at (18, 23).
18 hours after midnight is 6 pm.

4. x-intercepts are $x = -2, -1, 1, 3$ **[E]**

So factors are: $(x+2)(x+1)(x-1)(x-3)$

$(x+3)$ is not a factor so this eliminates A, B and C.

The graph 

is the shape of a negative quartic function which eliminates D and can be presented as

$y = -(x+2)(x+1)(x-1)(x-3)$

or $y = (x+2)(x+1)(x-1)(3-x)$

Can substitute $x = 0$ to check for a negative intercept.

5. Vertical asymptote at $x = 3$, so $b = -3$ **[C]**
Horizontal asymptote at $y = 2$, so $c = 2$
Eliminate A, D and E.

$$y = \frac{a}{x-3} + 2$$

Graph passes through (4, 0) substituting gives,

$$0 = \frac{a}{4-3} + 2$$

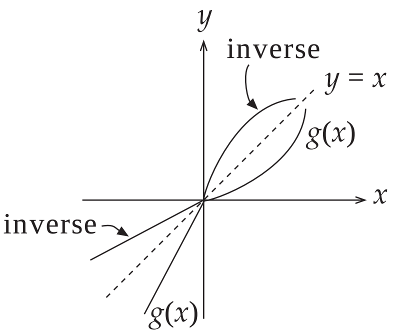
$$0 = a + 2$$

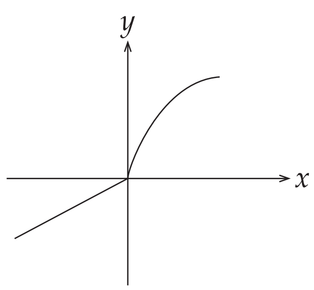
$$a = -2$$

6. $y = f(-x)$ gives a reflection about the y-axis. **[A]**

So the vertical asymptote will be reflected to become $x = -1$.

7. The graph of the inverse function is a reflection of $g(x)$ about the line $y = x$. **[A]**





8. Using the graphics calculator store the x values in L1 **[C]**

$\{1, 2, 3, 4, 5, 6, 7, 8\} \rightarrow$ L1

and the y values in L2

$\{1.6, 2.6, 4.3, \ldots..\} \rightarrow$ L2

Shape does not show linear, circular or logarithmic function, so eliminate A, D and E.

Either a power or exponential function.

To check which is the better fit press:

STAT CALC A: PwrReg ENTER

r = 0.9596 $r^2 = 0.9208$

STAT CALC 0: ExpReg ENTER

$r^2 = 0.9999$

Exponential has r^2 closer to 1.

9. $x^4 + x^3 - 3x^2 - 3x$ **[B]**

$= x(x^3 + x^2 - 3x - 3)$ common factor

$= x\left[x^2(x+1) - 3(x+1)\right]$ grouping

$= x(x+1)(x^2 - 3)$

$= x(x+1)(x+\sqrt{3})(x-\sqrt{3})$ DOPS

linear factors

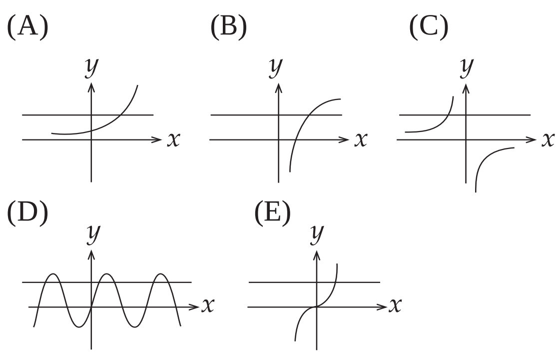
10. Let $m = 2^{\log_e(x+5)}$ **[A]**

Take the $\log_2$ of both sides

$\log_2 m = \log_2(x+5)$

so $m = x + 5$

11. A sketch of each function in the given domain: **[D]**

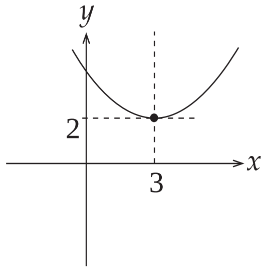


Shows that all functions except $f(x) = \sin x$ have one x value for every y value. i.e. are one-to-one.

For $f(x) = \sin x,\ x \in R$, the horizontal line test shows that there are many x values for the same y value, therefore not one-to-one.

12. For $f(x)$ to have an inverse function, it must be one–to–one. **[D]**

$f(x) = (x-3)^2 + 2$ is a many-to-one function if $x \in R$.



$f(x)$ is a quadratic function with turning point at (3, 2).

Use x values on one side of the turning point as the domain to create a one-to-one function.

Therefore the domain can be $x \le 3$

13. The graph of $f(x)$ yields the following observations for its gradient. **[C]**

when $x = -1$	steep positive gradient
when $x = 0$	positive gradient, steepness decreasing
when $x = 1$	positive gradient closer to 0
when $x \ge 2$	small, positive gradient approaching 0

14. $y = \log_e(\cos(2x))$ **[B]**

let $u = \cos 2x$, $\dfrac{du}{dx} = -2\sin 2x$

then $y = \log_e u$, $\dfrac{dy}{du} = \dfrac{1}{u}$

Using the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{u} \times -2\sin 2x\\ &= \frac{1}{\cos 2x} \times -2\sin 2x\\ &= \frac{-2\sin 2x}{\cos 2x}\\ &= -2\tan 2x\end{aligned}$$

15. Equation of normal $y = mx + c$ on curve $y = x\sin x$. **[D]**

Let $u = x$ and $v = \sin x$

$\therefore \dfrac{du}{dx} = 1$ $\qquad \therefore \dfrac{dv}{dx} = \cos x$

$y = uv$

$$\begin{aligned}\frac{d}{dx}(uv) &= u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{(product rule)}\\ &= x\cos x + \sin x \times 1\\ &= x\cos x + \sin x\end{aligned}$$

when $x = \pi$

$$\begin{aligned}y &= \pi \times \sin\pi\\ &= \pi \times 0\\ &= 0\end{aligned}$$

$(\pi, 0)$ point

$$\begin{aligned}\frac{dy}{dx} &= \pi\cos\pi + \sin\pi\\ &= \pi(-1) + 0\\ &= -\pi \ \text{ gradient of tangent}\end{aligned}$$

Gradient of normal

$m = \dfrac{-1}{-\pi} = \dfrac{1}{\pi}$

substitute $(\pi, 0)$ and $m = \dfrac{1}{\pi}$

in $y = mx + c$

$$\begin{aligned}0 &= \frac{1}{\pi} \times \pi + c\\ 0 &= 1 + c\\ c &= -1\end{aligned}$$

Equation $y = \dfrac{1}{\pi}x - 1$

$y = \dfrac{1}{\pi}(x - \pi)$

16. $\dfrac{dy}{dx} = -e^{-x}$ **[C]**

when $x = 0$

$$\begin{aligned}\frac{dy}{dx} &= -e^{-0}\\ &= -1\end{aligned}$$

17. $f(x+h) \simeq f(x) + hf'(x)$ **[A]**

$f(3.02) = f(3+0.02)$ where $x = 3$ $h = 0.02$

$\simeq f(3) + 0.02 f'(3)$

18. $f(0) = 0$ and $f(-3) = 0$ **[B]**

The function passes through the points (0, 0) and (–3, 0)

$f'(0) = 0$ and $f'(-1) = 0$.

There are stationary points at $x = 0$ and at $x = -1$

Therefore eliminate C.

$f'(x) > 0$ for $x < -1$

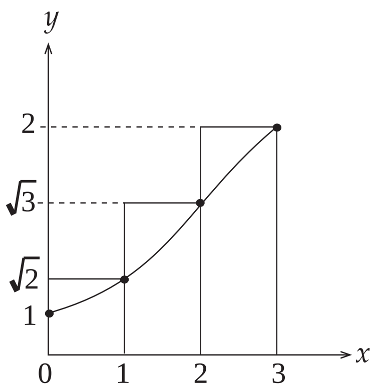
The function is increasing when $x < -1$, eliminate A and E.

$f'(x) < 0$ for $x > -1 \setminus \{0\}$

The function is decreasing for $x > -1 \setminus \{0\}$eliminate D.

19. $y = \sqrt{1+x}$ **[A]**

x	0	1	2	3
y	1	$\sqrt{2}$	$\sqrt{3}$	2



Area of rectangles

$= 1 \times \sqrt{2} + 1 \times \sqrt{3} + 1 \times 2$

$= \sqrt{2} + \sqrt{3} + 2$

20. $f'(x) = 2\cos(5x)$ **[B]**

antidifferentiating gives

$f(x) = \frac{2}{5}\cos(5x) + c$

To check differentiate answer.

21. $\frac{dy}{dx} = \frac{3}{(2x+1)^{\frac{1}{2}}}$ **[B]**

$\frac{dy}{dx} = 3(2x+1)^{-\frac{1}{2}}$

antidifferentiating gives

$$y = \frac{3(2x+1)^{\frac{1}{2}}}{\frac{1}{2} \times 2} + c$$

$$y = 3(2x+1)^{\frac{1}{2}} + c$$

22. Area from a to b: $\int_a^b f(x)dx$ **[D]**

Area from b to c:

$$\left|\int_b^c f(x)dx\right| = -\int_b^c f(x)dx = \int_c^b f(x)dx$$

Total Area $= \int_a^b f(x)dx + \int_c^b f(x)dx$

23. A discrete random variable is a 'counting' number (not a measurement). **[D]**

Goals are countable, you cannot have half a goal.

24. Let X be the number of \$10 chips drawn. **[A]**

Therefore X = 0, 1, 2, 3 or 4.

Hypergeometric distribution (without replacement).

N = 20, n = 4, D = 5

Pr(at least one \$10 chip)

$= \Pr(X \geq 1)$

$= 1 - \Pr(X = 0)$

$= 1 - \frac{{}^{5}C_0 \times {}^{15}C_4}{{}^{20}C_4}$

$= 1 - \frac{{}^{15}C_4}{{}^{20}C_4}$

25. $\mu = 10$ and $\sigma = 3$ [**A**]

For a binomial distribution $\mu = np$ and $\sigma = \sqrt{npq}$, $q = 1 - p$

$10 = np$ and $3 = \sqrt{npq}$

$9 = npq$

$9 = 10q$

$\therefore q = \frac{9}{10} = 0.9$

$p = 1 - q$

$p = 1 - 0.9$

$p = 0.1$

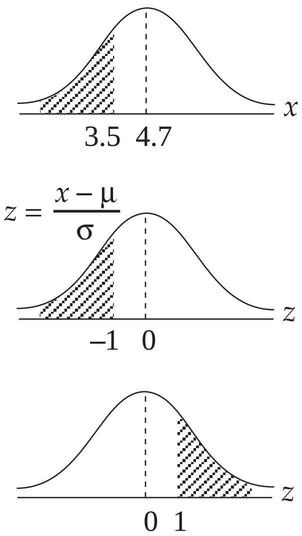
26. $X \sim N\left(4.7, 1.2^2\right)$ [**B**]

$\Pr(X < 3.5)$

$= \Pr\left(Z < \frac{3.5 - 4.7}{1.2}\right)$

$= \Pr(Z < -1)$

$= \Pr(Z > 1)$



27. $\mu = ?$ and $\sigma = 3$ [**E**]

$\Pr(X < 250) = 0.01$

$\Pr\left(Z < \frac{250 - \mu}{3}\right) = 0.01$

Using the graphics calculator,
press [2nd] [VARS] and choose
3: invNorm (0.01) ENTER

$\frac{250 - \mu}{3} = -2.326$

$250 - \mu = -6.979$

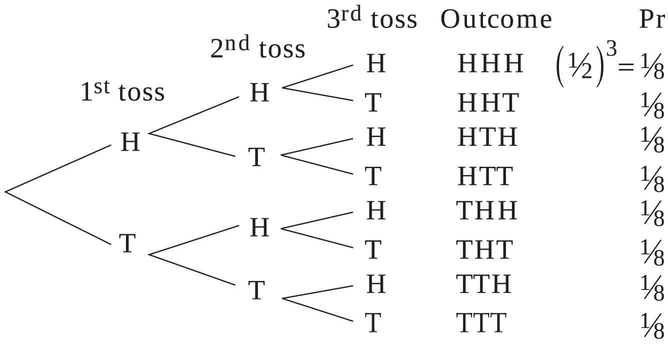
$\mu = 250 + 6.979$

$= 256.979$

$\simeq 257$

Part 2: Short-answers

1. a. Using a tree diagram:



$\Pr(X = 0) = \Pr(TTT) = \frac{1}{8}$

$\Pr(X = 1) = \Pr(TTH) + \Pr(THT) + \Pr(HTT) = \frac{3}{8}$

$\Pr(X = 2) = \Pr(THH) + \Pr(HTH) + \Pr(HHT) = \frac{3}{8}$

$\Pr(X = 3) = \Pr(HHH) = \frac{1}{8}$

OR Using the Binomial Distribution: $X \sim Bi\left(3, \frac{1}{2}\right)$

$\Pr(X = 0) = \left({}^3C_0\right) \times \left(\frac{1}{2}\right)^0 \times \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$\Pr(X = 1) = \left({}^3C_1\right) \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = 3 \times \left(\frac{1}{2}\right)^3 = \frac{3}{8}$

$\Pr(X = 2) = \left({}^3C_2\right) \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^1 = 3 \times \left(\frac{1}{2}\right)^3 = \frac{3}{8}$

$\Pr(X = 3) = \left({}^3C_3\right) \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

OR Using the gaphics calculator,

[2nd] [VARS] 0: binompdf $\left(3, \frac{1}{2}\right)$

will list the probability distribution.
Answer can be converted to fractions using MATH 1: Frac

X	$\Pr(X = x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

b. $E(X) = \sum x p(x)$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$
$$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$
$$= \frac{12}{8}$$
$$= 1.5$$

OR

$E(X) = np$ for binomial distribution

$$= 3 \times \frac{1}{2}$$
$$= 1.5$$

2. Without replacement therefore Hypergeometric.

N = 100, n = 5, D = 5

Let X represent the number of defective alarms in the sample,

X = 0, 1, 2, 3, 4, 5

Accepted when X = 0

$$\Pr(X = 0) = \frac{\left({}^{5}C_0\right)\left({}^{95}C_5\right)}{\left({}^{100}C_5\right)}$$
$$= 0.770 \quad \text{(3 decimal places)}$$

Note: there are calculator programs that are available which can calculate this.

3. a. Period = 8

b. Amplitude $= \dfrac{1.5 + 0.5}{2} = 1.0$

4. a. $2 \times 2^{-2x} = 2002$

$2^{-2x} = 1001$ $\quad$ ($\div 2$ both sides)

$\log_e 2^{-2x} = \log_e 1001$ $\quad$ ($\log_e$ both sides)

$-2x \log_e 2 = \log_e 1001$ ($\div -2 \log_e 2$ both sides)

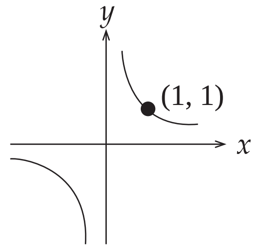
$$x = \frac{\log_e 1001}{-2 \log_e 2}$$

$x = -4.984$ $\quad$ (3 decimal places)

b. $2 \log_e(3x + 1) - \log_e(x)$

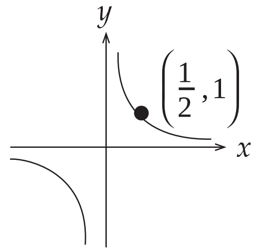
$$= \log_e(3x+1)^2 - \log_e(x)$$
$$= \log_e \frac{(3x+1)^2}{x}$$

5. a.



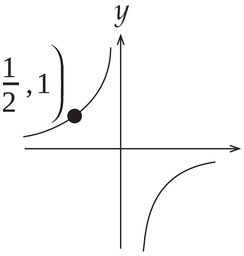
$y = \dfrac{1}{x}$

dilation by a factor of $\frac{1}{2}$ from y-axis



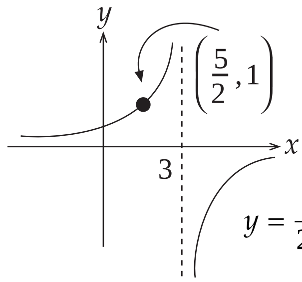
$\left(\frac{1}{2}, 1\right)$ $\quad$ $y = \dfrac{1}{2x}$

reflection in the y-axis



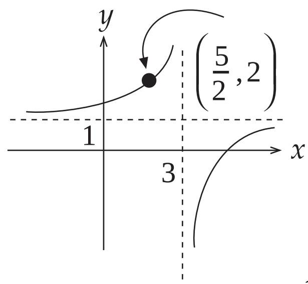
$\left(-\frac{1}{2}, 1\right)$ $\quad$ $y = \dfrac{-1}{2x}$

translation +3 units parallel to the x-axis



$\left(\frac{5}{2}, 1\right)$ $\quad$ $y = \dfrac{-1}{2(x-3)}$

translation of +1 unit parallel to the y-axis



$\left(\frac{5}{2}, 2\right)$ $\quad$ $y = \dfrac{-1}{2(x-3)} + 1$

b. domain = R \ {3}

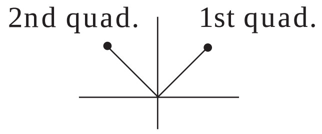
range = R \ {1}

6. a. $2\sin(3x) - 1 = -0.5 \qquad x \in \left[0, \frac{\pi}{2}\right]$

$2\sin(3x) = 0.5$

$\sin(3x) = 0.25$

$3x = \sin^{-1}(0.25)$



$3x = 0.253, \pi - 0.253 \qquad 3x \in \left[0, \frac{3\pi}{2}\right]$

$x = \frac{0.253}{3}, \frac{\pi - 0.253}{3}$

$x = 0.084, 0.963$ (3 decimal places)

OR Using graphics calculator in RADIAN mode.

Enter $y_1 = 2\sin(3x) - 1$

$y_2 = -0.5$

WINDOW Xmin = 0

Xmax = $\frac{\pi}{2}$

ZOOMFIT

Graph and find points of intersection

Press [2nd] [TRACE] 5: Intersect

Answer $x = 0.084, 0.963$ (3 decimal places)

b. $f(x) = 2\sin(3x) - 1$

$f'(x) = 6\cos(3x)$

When $x = 1 \qquad f'(x) = 6\cos(3)$

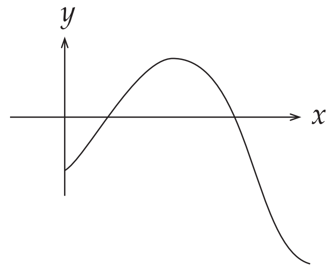
$= -5.940$ (3 d.p.)

OR Using graphics calculator

Press [2nd] [TRACE] 6: $\frac{dy}{dx}$ $x = 1$ [ENTER]

$\frac{dy}{dx} = -5.940$ (3 decimal places)

c.



The rate of change is positive to the left of the turning point as shown on the graph.

Find the turning point:

Press [2nd] [TRACE] 4: Maximum etc.

This gives an x-coordinate of 0.523

The interval over which the rate of change is positive is (0, 0.523) (3 decimal places)

OR To find the turning point:

$f'(x) = 0$

$6\cos(3x) = 0 \qquad x \in \left[0, \frac{\pi}{2}\right]$

$\cos(3x) = 0 \qquad 3x \in \left[0, \frac{3\pi}{2}\right]$

$3x = \cos^{-1}(0)$

$3x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{6}, \frac{3\pi}{6}$

x	$< \pi/6$	$\pi/6$	$> \pi/6$
$f'(x)$	$+ve$	0	$-ve$
	/	—	\

Rate is positive when $x \in \left(0, \frac{\pi}{6}\right)$ i.e.

$x \in (0, 0.524)$ (3 decimal places)

7. a. $f'(x)$ does not exist at the points of discontinuity i.e. at $x = -2, 0, 4$.

$f'(x)$ does not exist where curves are not smooth i.e. at $x = 2$.

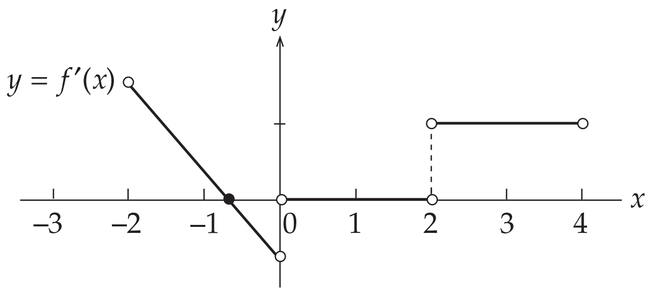
$f'(x) = 0$ when $x \simeq -0.6$ and when $0 < x < 2$.

$f'(x) > 0$ when $-2 < x < -0.6$ and when $2 < x < 4$.

$f'(x) < 0$ when $-0.6 < x < 0$ and becomes more negative from left to right.

For $x \in (-2, -0.6)$, the positive gradient decreases from left to right approaching zero.

For $2 < x < 4$ $f'(x) = \frac{\text{rise}}{\text{run}} = \frac{2}{2} = 1$



b. Dom $f' = (-2,0) \cup (0,2) \cup (2,4)$

8. a. Algebraic method, solve simultaneously

$$2x^2 + 4x - 5 = 3x + 1$$

$$2x^2 + x - 6 = 0$$

$$(2x - 3)(x + 2) = 0$$

$$2x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = \frac{3}{2},\ x = -2$$

OR Using graphics calculator

$y_1 = 2x^2 + 4x - 5$

$y_2 = 3x + 1$

Zoomstandard

Find intersection by pressing

[2nd] [TRACE] 5: Intersect etc.

$x = \frac{3}{2}$ and $x = -2$

b. When $x \in \left[-2, \frac{3}{2}\right]$

$y_2 > y_1$ where $y_1 = 2x^2 + 4x - 5$

and $y_2 = 3x + 1$

$$\text{Area} = \int_{-2}^{\frac{3}{2}} (3x + 1) - (2x^2 + 4x - 5)dx$$

$$= \int_{-2}^{\frac{3}{2}} \left(3x + 1 - 2x^2 - 4x + 5\right)dx$$

$$= \int_{-2}^{\frac{3}{2}} (-2x^2 - x + 6)dx$$

$$= \left[\frac{-2x^3}{3} - \frac{x^2}{2} + 6x\right]_{-2}^{\frac{3}{2}}$$

$$= \left(\frac{-2\left(\frac{3}{2}\right)^3}{3} - \frac{\left(\frac{3}{2}\right)^2}{2} + 6\left(\frac{3}{2}\right)\right) - \left(\frac{-2(-2)^3}{3} - \frac{(-2)^2}{2} + 6(-2)\right)$$

$$= \left(\frac{-27}{9} - \frac{9}{8} + 9\right) - \left(\frac{16}{3} - 2 - 12\right)$$

$$= \frac{45}{8} + \frac{26}{3}$$

$\simeq 14.292$ square units

