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MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 2002 SOLUTIONS

Part I – Multiple-choice answers

1. D	8. E	15. B	22. C
2. A	9. C	16. C	23. A
3. D	10. C	17. E	24. B
4. E	11. B	18. C	25. B
5. E	12. B	19. D	26. E
6. A	13. D	20. D	27. D
7. E	14. E	21. B	

Part I – Multiple-choice solutions

Question 1

The vertical distance between a minimum and maximum point on the function is 3. The amplitude of the function is therefore $\frac{3}{2}$.

The answer is D.

Question 2

The range of the function is $y \in [-5,1]$ and so the amplitude is 3. The period of the function is $5-1=4$. So $\frac{2\pi}{n}=4$. So, $n=\frac{\pi}{2}$.

There is a horizontal shift of 1 unit to the right.
There is a vertical shift of 2 units down.

The required equation is $y=3\cos\frac{\pi}{2}(x-1)-2$.

The answer is A.

Question 3

If $a=1$, option A would be the correct graph.

If $a=\frac{\pi}{2}$, option B would be the correct graph.

If $a=-1$, option C would be the correct graph.

If $a=\frac{1}{4}$, option E would be the correct graph.

The graph shown in option D requires a horizontal shift ie. $y=\tan\left(x-\frac{\pi}{2}\right)$.

No value of a will achieve this.
The answer is D.

Question 4

If $b=-a$, then the graph of $y=b$ becomes $y=-a$ and the straight line is a tangent to the function *f* and hence $a\sin(2x)=b$ will have one solution.
Similarly, this will happen if $b=a$. So options A, C and D can be eliminated. If $b=0$, then the equation $y=b$ becomes $y=0$ which is the equation of the x-axis and this intersects with the graph of function *f* on 3 occasions. Hence there are 3 solutions to the equation $a\sin(2x)=0$.

If $b = a + 1$, then the graph of $y = b$ becomes $y = a + 1$ which is a horizontal line one unit above the maximum point on the graph of $y = f(x)$. Hence $y = a + 1$ does not intersect with $y = f(x)$. Hence $a\sin(2x) = a + 1$ has no solutions. The answer is E.

Question 5

The required term is given by ${}^8C_4(x)^4(1)^4 = 70x^4$

The coefficient is 70. The answer is E.

Question 6

$$\begin{aligned}
&\log_2 \frac{1}{4} - \log_2 32 + 2\log_2 1 \\
&= \log_2\left(\frac{1}{4} \div 32\right) + 2 \times 0 \\
&= \log_2 \frac{1}{128} \\
&= \log_2\left(\frac{1}{2^7}\right) \\
&= \log_2 2^{-7} \\
&= -7\log_2 2 \\
&= -7
\end{aligned}$$

The answer is A.

Question 7

$$\begin{aligned}
4 \times 3^{3x+1} &= 24 \\
3^{3x+1} &= 6 \\
\log_{10} 3^{3x+1} &= \log_{10} 6 \\
(3x+1)\log_{10} 3 &= \log_{10} 6 \\
3x + 1 &= \frac{\log_{10} 6}{\log_{10} 3} \\
x &= \frac{1}{3}\left(-1 + \frac{\log_{10} 6}{\log_{10} 3}\right)
\end{aligned}$$

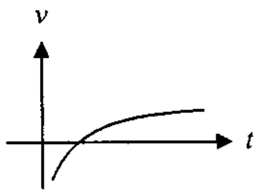
The answer is E.

Question 8

The graph of option A crosses once.
The graph of option B touches twice.
The graph of option C crosses 3 times.
The graph of option D crosses 3 times.
The graph of option E crosses twice.
The answer is E.

Question 9

The shape of the graph produced by the data is best described as logarithmic.
The answer is C.



Question 10

The domain of f is R.
The range of f is $(-\infty,4]$.
The answer is C.

Question 11

A function will have an inverse function if it passes the horizontal line test. That is if a horizontal line can be drawn anywhere on the graph and be cut by the function no more than once. Only the function in option B would fail the horizontal line test. That is, a horizontal line drawn would cut the function twice.
The answer is B.

Question 12

The graph of $y=\frac{2}{x+1}$ has asymptotes with equations $x=-1$ and $y=0$.
The graph of $y=e^{x+1}$ has an asymptote with equation $y=0$.
The graph of $y=\log_e(x-1)$ has asymptote with equation $x=1$.
The graph of $y=x^{-2}$ has asymptotes with equations $x=0$ and $y=0$.
The graph of $y=\sqrt{x-1}$ has no asymptotes.
The answer is B.

Question 13

$f(x+h)\approx f(x)+hf'(x)$ and $f(x)=\frac{1}{x}$.

Now $\frac{1}{10.01}=\frac{1}{10+0.01}$
So $f(10+0.01)\approx f(10)+0.01f'(10)$
The answer is D.

Question 14

Method 1 Expand first

Let $y=\sqrt{x}(x^2+1)$

$$=x^{\frac{1}{2}}(x^2+1)$$

$$=x^{\frac{5}{2}}+x^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{5}{2}x^{\frac{3}{2}}+\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{5}{2}x^{\frac{3}{2}}+\frac{1}{2\sqrt{x}}$$

The answer is E.

Method 2 Product rule

$$y=x^{\frac{1}{2}}(x^2+1)$$

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}(x^2+1)+x^{\frac{1}{2}}\times 2x$$

$$=\frac{x^2+1}{2\sqrt{x}}+2x^{\frac{3}{2}}$$

$$=\frac{x^2}{2\sqrt{x}}+\frac{1}{2\sqrt{x}}+2x^{\frac{3}{2}}$$

$$=\frac{1}{2}x^{\frac{3}{2}}+\frac{1}{2\sqrt{x}}+2x^{\frac{3}{2}}$$

$$=\frac{5}{2}x^{\frac{3}{2}}+\frac{1}{2\sqrt{x}}$$

Question 15

$$y = \log_e(\sin(2x))$$

Method 1

Let $y = \log_e u$ where $u = \sin(2x)$

So, $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2\cos(2x)$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ Chain rule

$$= \frac{1}{u} \cdot 2\cos(2x)$$

$$= \frac{2\cos(2x)}{\sin(2x)}$$

$$= \frac{2}{\tan(2x)}$$

Note that $\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$

So, $\frac{1}{\tan(2x)} = \frac{\cos(2x)}{\sin(2x)}$

Method 2

$$y = \log_e(\sin(2x))$$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sin(2x))}{\sin(2x)}$$

$$= \frac{2\cos(2x)}{\sin(2x)}$$

$$= \frac{2}{\tan(2x)}$$

The answer is B.

Question 16

$$g(x) = x^5 e^{2x}$$

So $g'(x) = x^5 \times 2e^{2x} + 5x^4 e^{2x}$

$$= 2x^5 e^{2x} + 5x^4 e^{2x}$$

So $g'(1) = 2e^2 + 5e^2$

$$= 7e^2$$

The answer is C.

Question 17

$$h(x)=\frac{x}{\sqrt{x+1}}$$

$$h'(x)=\frac{\sqrt{x+1}\cdot 1-\frac{1}{2}(x+1)^{-\frac{1}{2}}\times x}{(\sqrt{x+1})^2} \qquad \text{(quotient rule)}$$

$$=\left(\sqrt{x+1}-\frac{x}{2\sqrt{x+1}}\right)\div(x+1)$$

$$=\frac{2\sqrt{x+1}\sqrt{x+1}-x}{2\sqrt{x+1}}\times\frac{1}{x+1}$$

$$=\frac{2(x+1)-x}{2\sqrt{x+1}}\times\frac{1}{x+1}$$

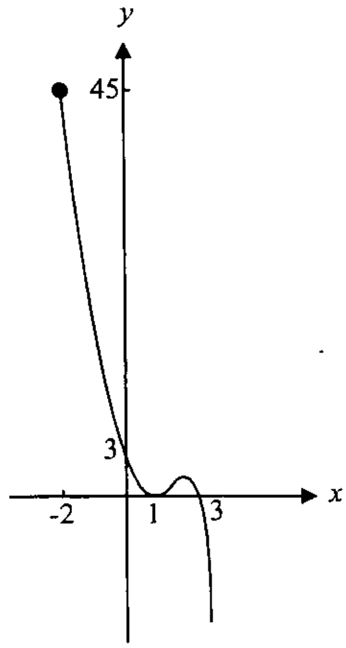
$$=\frac{2x+2-x}{2\sqrt{x+1}(x+1)}$$

$$=\frac{x+2}{2\sqrt{x+1}(x+1)}$$

The answer is E.

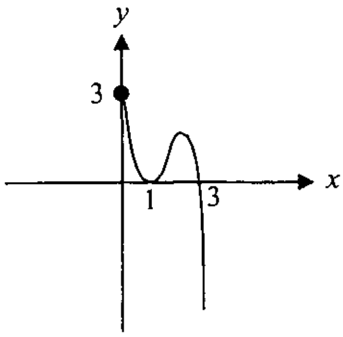
Question 18

Look at the graph.
For $a=-2$, the graph is shown.



In this case the maximum value of f is 45 and this occurs at $x=-2$.

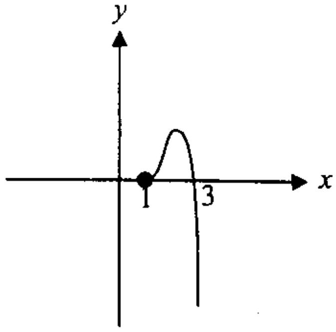
For $a=0$, the graph is shown.



In this case the maximum value of f occurs when $x=0$.

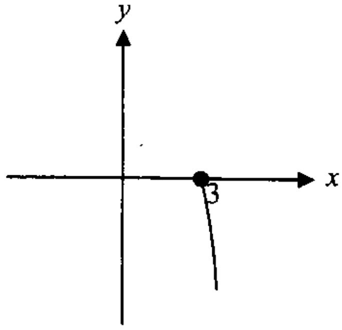
For $a = 1$, the graph is shown.

In this case, the maximum value of f occurs at $x = \frac{7}{3}$.



For $a = 3$, the graph is shown.

The maximum value occurs at $x = 3$.



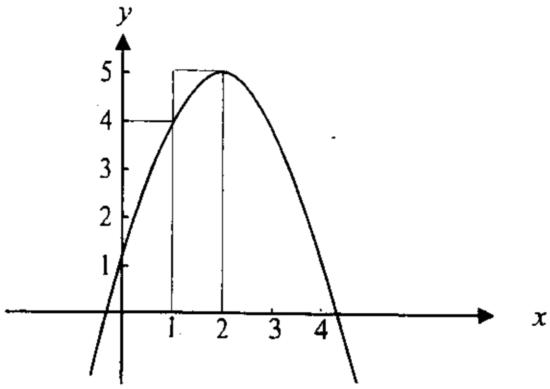
And similarly for $a = 5$, the graph would be well below the x-axis and the maximum value negative.
The answer is C.

Question 19

From the diagram, we have

$$\begin{aligned}\text{Area required} &= f(1)\times 1 + f(2)\times 1\\ &= 4 + 5\\ &= 9 \text{ square units.}\end{aligned}$$

The answer is D.



Question 20

$$\int\left(2\sqrt{x} - e^{-2x}\right)dx = \int\left(2x^{\frac{1}{2}} - e^{-2x}\right)dx$$

$$= 2x^{\frac{3}{2}} \times \frac{2}{3} + \frac{e^{-2x}}{2} + c$$

$$= \frac{4x^{\frac{3}{2}}}{3} + \frac{e^{-2x}}{2} + c$$

The answer is D.

Question 21

Method 1

$$\int_0^k (3x+2)^5\,dx = \left[\frac{1}{3\times 6}(3x+2)^6\right]_0^k$$

$$= \frac{1}{18}\left\{(3k+2)^6 - 2^6\right\}$$

So $\frac{1}{18}\left\{(3k+2)^6 - 64\right\} = 864.5$

$$(3k+2)^6 - 64 = 15561$$

$$(3k+2)^6 = 15625$$

$$3k+2 = \pm\sqrt[6]{15625}$$

$$3k+2 = \pm 5$$

$$k = -\frac{7}{3} \text{ or } 1 \text{ but } k>0 \text{ so } k=1$$

Method 2

On your graphics calculator, graph $y=(3x+2)^5$.

Use 2nd calc and $\int f(x)dx$.

The lower limit is zero. Substitute values for k, ie. the upper limit, until $\int f(x)dx = 864.5$

The answer is B.

Question 22

Option A - $f'(a)=0$, so the gradient of the function f at $x=a$ is zero.

Option B – for $x<0$, $f'(x)>0$, so the gradient of f is greater than zero, that is, positive.

Option C – $f'(b)\neq 0$ and hence there is not a stationary point at $x=b$. This option is incorrect.

Options D and E are both correct.

The answer is C.

Question 23

$$\text{variance of } X = E(X^2) - [E(X)]^2$$

$$= (0^2\times 0\cdot 1 + 1^2\times 0\cdot 3 + 2^2\times 0\cdot 4 + 3^2\times 0.2) - (0\times 0\cdot 1 + 1\times 0\cdot 3 + 2\times 0\cdot 4 + 3\times 0\cdot 2)^2$$

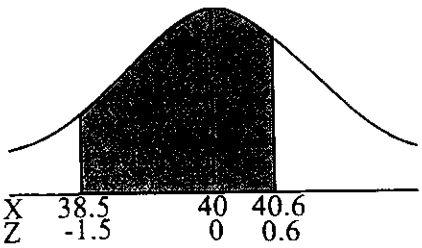
$$= 0\cdot 3 + 1\cdot 6 + 1\cdot 8 - (1\cdot 7)^2$$

$$= 0.81$$

The answer is A.

Question 24

Method 1



$\Pr(38\cdot 5 < X < 40\cdot 6) = \Pr(-1.5 < Z < 0.6)$

since $Z = \dfrac{X-\mu}{\sigma}$

so, $Z = \dfrac{38\cdot 5 - 40}{1} = -1.5$ and $Z = \dfrac{40\cdot 6 - 40}{1} = 0.6$

So $\Pr(-1\cdot 5 < Z < 0\cdot 6) = \Pr(Z<0.6) - \Pr(Z<-1.5)$

$$= \Pr(Z<0\cdot 6) - (1 - \Pr(Z<1\cdot 5))$$

$$= 0\cdot 7257 - 1 + 0\cdot 9332$$

$$= 0\cdot 6589$$

Method 2 Using a graphics calculator

2nd DISTR

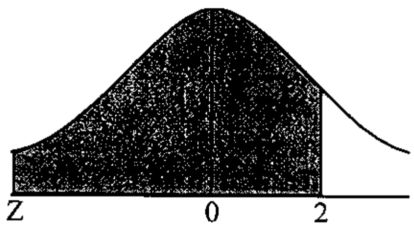
normalcdf(38.5,40.6,40,1)

=0.6589

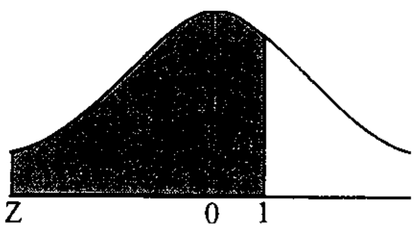
The answer is B.

Question 25

Z is the standard normal variable and so $\mu = 0$ and $\sigma = 1$.
So, option A is eliminated and option B is correct.
Note that $\Pr(-1 < Z < 1) = 0.68$ so option C is incorrect.
$\Pr(Z < 0) = 0.5$ whereas $1 - 2\Pr(Z > 0) = 1 - 2 \times 0 \cdot 5 = 0$ so option D is incorrect.
$\Pr(Z < 2)$ is shown as the shaded region below.



$\Pr(Z < 1)$ is shown as the shaded region below.



Clearly twice this shaded area does not equal the shaded area shown in the previous diagram.
The answer is B.

Question 26

The number of times that Colin is late for work over the five mornings from Monday to Friday is a random variable with a binomial distribution with $n = 5$ and $p = 0.3$.

So mean $= np = 1.5$ and variance $= np(1-p)$

$$= 5 \times 0.3 \times 0.7$$
$$= 1.05$$

The answer is E.

Question 27

We have a binomial distribution with $n = 10$ and $p = 0.6$.

$\Pr(\text{at least 8 complete course}) = \Pr(X = 8) + \Pr(X = 9) + \Pr(X = 10)$

$= {}^{10}C_8(0.6)^8(0.4)^2 + {}^{10}C_9(0.6)^9(0.4)^1 + {}^{10}C_{10}(0.6)^{10}(0.4)^0$

$= {}^{10}C_8(0.6)^8(0.4)^2 + 4(0.6)^9 + (0.6)^{10}$

The answer is D.

PART II

Question 1

a.

$$f(x) = -x^3 + 7x^2 - 11x + 5$$
$$f(5) = -125 + 175 - 55 + 5$$
$$= 0$$

So $x - 5$ is a factor. **(1 mark)**

$$f(x) = -x^2(x-5) + 2x(x-5) - 1(x-5)$$
$$= (x-5)(-x^2 + 2x - 1)$$
$$= -(x-5)(x^2 - 2x + 1)$$
$$= -(x-5)(x-1)^2$$

(1 mark)

b. Since $(x-1)$ is a repeated factor, the graph touches the x-axis at $x = 1$. So there is a turning point at $(1,0)$.

(1 mark)

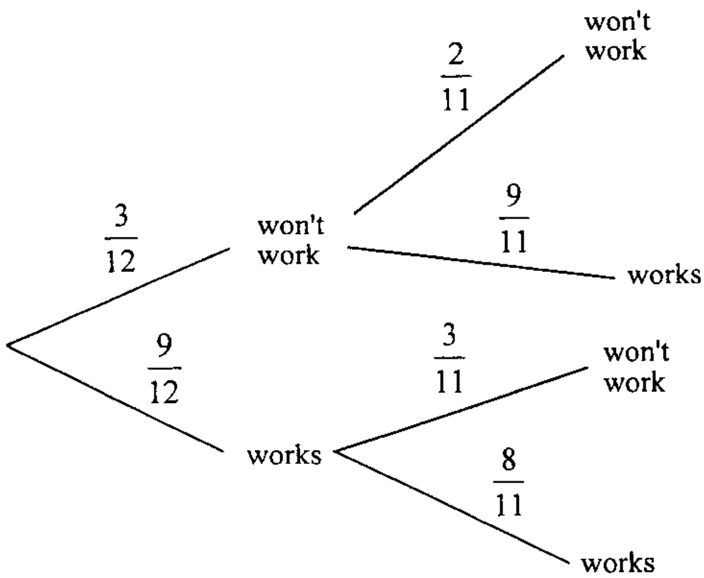
Question 2

a.

$$\mathrm{E}(\mathrm{X}) = \frac{nD}{N}$$
$$= \frac{2 \times 3}{12}$$
$$= 0.5$$

(1 mark)

b. Method 1



(1 mark)

$$\Pr(\text{both won't work}) = \frac{3}{12} \times \frac{2}{11} = \frac{1}{22}$$

(1 mark)

Method 2

$$\Pr(X=2)=\frac{{}^{3}C_{2}\,{}^{(12-3)}C_{(2-2)}}{{}^{12}C_{2}}$$

$$=\frac{3\times{}^{9}C_{0}}{{}^{12}C_{2}}$$

$$=\frac{3}{66}$$

$$=\frac{1}{22}$$

(2 marks)

Question 3

$$\sin(3x)+2\cos(3x)=0$$

$$\sin(3x)=-2\cos(3x)$$

$$\frac{\sin(3x)}{\cos(3x)}=-2$$

$$\tan(3x)=-2$$

(1 mark)

Now, $0^\circ \le x \le 180^\circ$

So, $0^\circ \le 3x \le 540^\circ$

So, since $\tan(3x)=-2$

S	A
T	C

(Check that your calculator is in "Degree" mode.)

$$3x=180^\circ-63.4349^\circ,\ 360^\circ-63.4349^\circ,\ 540^\circ-63.4349^\circ$$

$$3x=116.5651^\circ,\ 296.5651^\circ,\ 476.5651^\circ$$

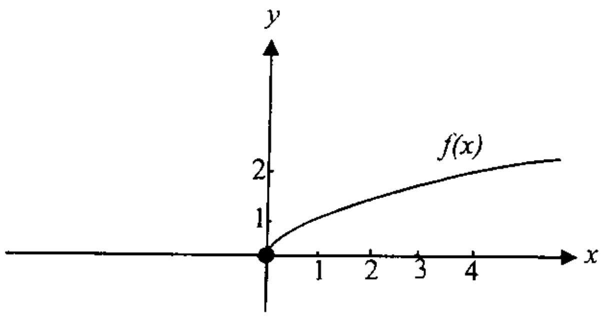
$$x=38.8550^\circ,\ 98.8550^\circ,\ 158.8550^\circ$$

$$x=38^\circ 51',\ 98^\circ 51',\ 158^\circ 51'$$

(1 mark)

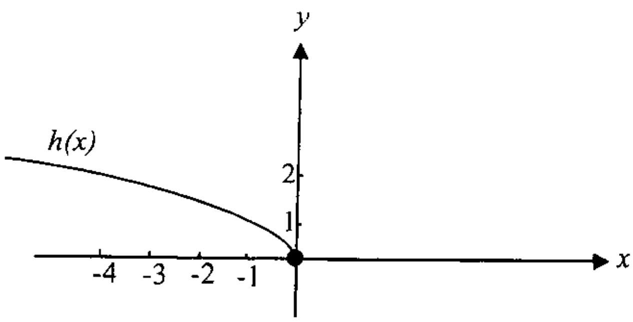
Question 4

a.



(1 mark)

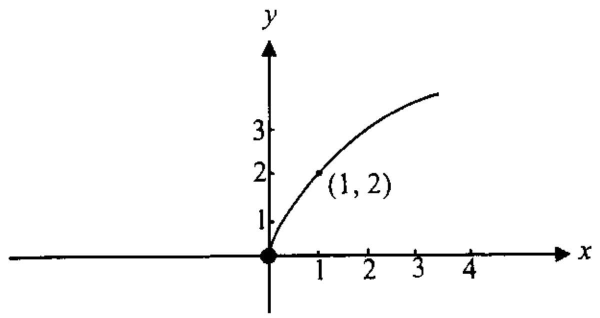
b. (i)



(1 mark)

(ii) $h(x)=\sqrt{-x}$ **(1 mark)**

c.



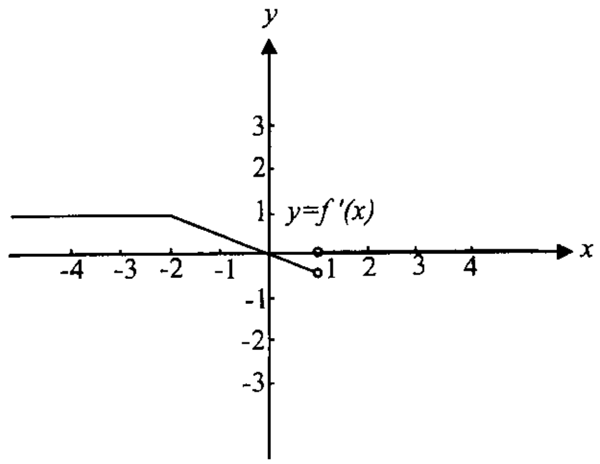
(1 mark)

Question 5

a. $d_f = R, \quad r_f = (-\infty, 1]$

(1 mark)

b.



Note that the graph of $y=f(x)$ is "smoothly continuous" at $x=-2$ and hence $f'(x)$ exists there but the graph of $y=f(x)$ is not continuous at the point where $x=1$ and hence $f'(x)$ does not exist there.

(2 marks)

Question 6

$$y = 2\tan(\frac{x}{4})+1$$

$$\frac{dy}{dx}=\frac{2}{4}\sec^2(\frac{x}{4})$$
$$=\frac{1}{2}\sec^2(\frac{x}{4})$$

(1 mark)

When $x=\pi$, $\frac{dy}{dx}=\frac{1}{2}\sec^2(\frac{\pi}{4})$

$$=\frac{1}{2}\times\frac{1}{\cos^2\left(\frac{\pi}{4}\right)}$$

$$=\frac{1}{2}\times\frac{1}{(\frac{1}{\sqrt{2}})^2}$$

$$= \frac{1}{2} \times \frac{1}{\frac{1}{2}}$$

$$= \frac{1}{2} \times 2$$

$$= 1$$

So the gradient of the tangent at $x = \pi$ is 1 and hence the gradient of the normal at $x = \pi$ is -1. **(1 mark)**

When $x = \pi$, $y = 2\tan(\frac{\pi}{4}) + 1$

$= 3$ **(1 mark)**

So the normal to the curve at the point $(\pi, 3)$ is given by

$$y - 3 = -1(x - \pi)$$

$$y = -x + \pi + 3$$

(1 mark)

Question 7

Area required

$$= \int_2^4 \left(e^{\frac{x}{2}} - e - \cos\left(\frac{\pi x}{4}\right)\right) dx + \int_0^2 \left(\cos\left(\frac{\pi x}{4}\right) - \left(e^{\frac{x}{2}} - e \right)\right) dx$$ **(1 mark)**

$$= \left[2e^{\frac{x}{2}} - ex - \frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right)\right]_2^4 + \left[\frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right) - 2e^{\frac{x}{2}} + ex \right]_0^2$$ **(1 mark)**

$$= \left\{ \left(2e^2 - 4e - \frac{4}{\pi}\sin \pi \right) - \left(2e - 2e - \frac{4}{\pi}\sin\left(\frac{\pi}{2}\right)\right)\right\} + \left\{ \left(\frac{4}{\pi}\sin\left(\frac{\pi}{2}\right) - 2e + 2e \right) - \left(\frac{4}{\pi}\sin 0 - 2e^{\circ} + 0 \right)\right\}$$

(1 mark)

$$= 2e^2 - 4e - 0 + \frac{4}{\pi} + \frac{4}{\pi} + 2$$

$$= 2e^2 - 4e + 2 + \frac{8}{\pi}$$

(1 mark)

Total 23marks