Mathematical Methods (CAS) pilot study: supplementary questions – extended response

Question 1

The volume of water, w megalitres, at time t months in a dam is modelled by the formula

$$w(t) = 4 - 0.5\sin\left(\frac{\pi t}{3}\right) - \cos\left(\frac{\pi t}{6}\right), \text{ where } t \geq 0$$

and t is measured from the first of January.

a. Find the volume of water in the dam on January 1. 1 mark

b. Show that the period of w is 12. 2 marks

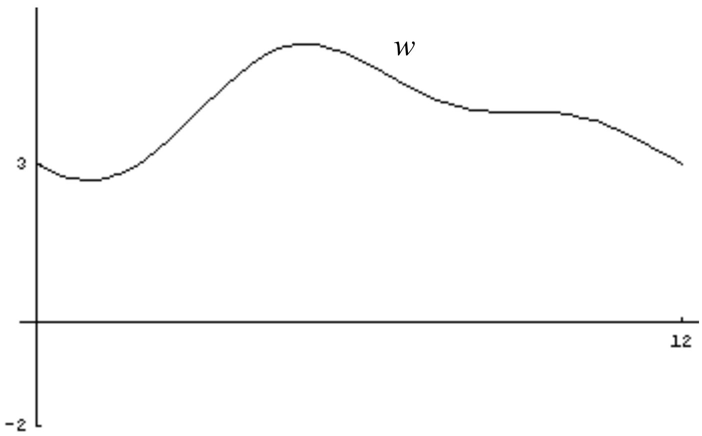
c. Determine the average volume of water in the dam over a year. 2 marks

d. **i.** Find the exact rate of change of volume, with respect to time, when $t = 2$ 2 marks

ii. For what values of t is the rate of change of volume, with respect to time, zero for $t \in [0,12]$? 2 marks

iii. Find the exact values of the maximum and minimum volume, for $t \in [0,12]$. 2 marks

e. Part of the graph of w is shown below



On the same set of axes sketch the corresponding part of the graph of $y = \frac{dw}{dt}$, given that the maximum positive gradient of w is approximately 1.

2 marks

Total: 13 marks

Question 2

A guidance system contains a special electronic switch whose time in years to failure is given by the random variable X, with probability density function

$$f(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{5} e^{-\frac{x}{5}} & x > 0 \end{cases}$$

a. Find the exact value of the mean time to failure for one of these special electronic switches. 2 marks

b. Find the median time, correct to 2 decimal places of a year, to failure for one of these special electronic switches. 2 marks

c. Find the variance of the time to failure for one of these special electronic switches. 2 marks

d. Find the probability that one of these special switches will last for at least 5 years. 2 marks

Four of these switches are installed on a satellite. The satellite can continue operating if at least two of the switches have not failed.

e. What is the expected number of switches in the satellite that will last for at least 5 years? 2 marks

f. What is the probability that the satellite will still be operational after 5 years? 2 marks

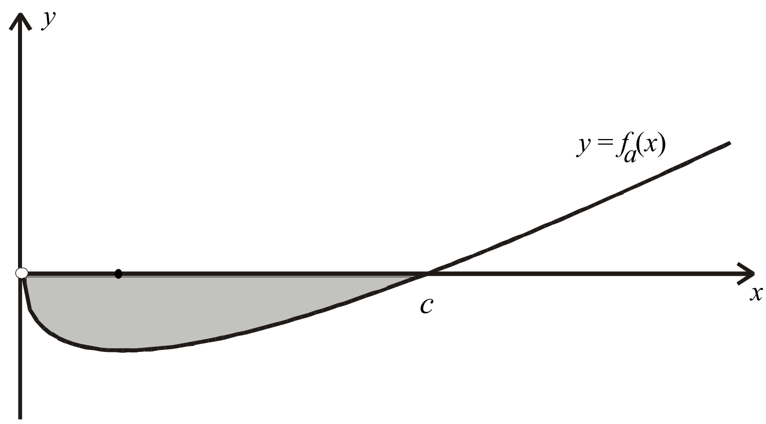
Total: 12 marks

Question 3

Consider the family of functions f_a: $[0, \infty) \rightarrow R$, defined by

$f_a(x) = x - a\sqrt{x}$

where a is a real number, $a > 0$. Part of the graph of f_a is shown below.



a. Find c in terms of a, where $f_a(c) = 0$ and c is not zero. 2 marks

b. Determine intervals on which f_a is a decreasing function and the intervals on which f_a is an increasing function.

4 marks

c. Find the equation to the tangent to the graphs of f_a at the point (c, 0).
What can be said about the family of such tangents? 3 marks

d. What is the range of f_a ? 2 marks

e. Find the exact value of the area of the shaded region in terms of a. 2 marks

f. Let $g_a : (b, \infty) \rightarrow$ R be a function with the same rule as f_a, where b is the least value of x such that g_a has an inverse function.

i. Find b in terms of a. 1 mark

ii. Find the rule for g_a^{-1}, the inverse function of g_a. 3 marks

iii. What is the domain of g_a^{-1} ? 1 mark

Total: 18 marks

Question 4

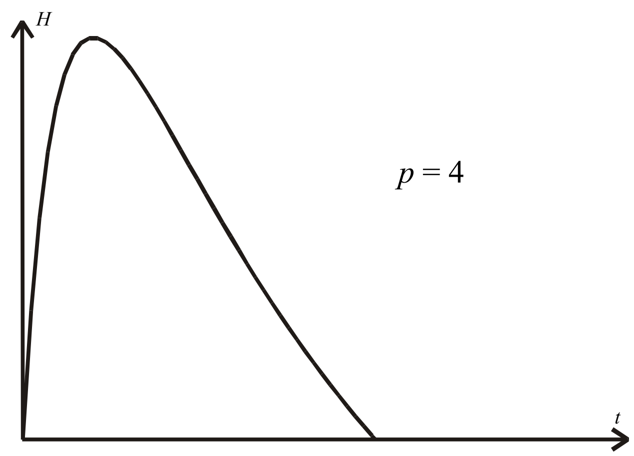
The chemical coprazide is manufactured in a large vat. The raw products are mixed in the vat and immediately start to react to form a substance called hydrocoprazine. The hydrocoprazine decomposes to form coprazide. Since hydrocoprazine is explosive in high concentrations, the process must be monitored carefully so that the concentration of hydrocoprazine is kept at a safe level.

The quantity of hydrocoprazine in the vat at time t minutes after the process starts may be modelled using the formula

$$H = \frac{-pt(pt - 12)}{(t + 1)^2} \quad \text{where } p > 0$$

and H Kg is the quantity of hydrocoprazine present at time t minutes and $10p$ ºC is the temperature at which the vat is maintained during the reaction. The process is finished when the quantity of hydrocoprazine becomes zero.

a. The graph of H when $p = 4$ is shown. Sketch graphs of H for $p = 2$ and 6 on the same axes. 2 marks



b. Find an expression for the time taken for the process as a function of p.

2 marks

Question 4 continued next page

c. **i.** Find an equation in terms of p and t, that when solved for t will give an expression for the time at which the maximum quantity of hydrocoprazine is present in the vat as a function of p.

2 marks

ii. Solve this equation to give this time as a function of p. 2 marks

d. Hence find an expression in terms of p for the maximum quantity of hydrocoprazine that is present in the vat during the reaction. 2 marks

e. For safety reasons, the quantity of hydrocoprazine in the vat must always be less than 12 kg. Find the greatest value of K such that if $p \leq K$ then $H \leq 12$.

2 marks

f. If p is equal to the value for K found in **e.**, find

i. the temperature at which the reaction must take place 1 mark

ii. the time taken for the reaction to finish. 1 mark

Total: 14 marks

Question 5

It is proposed to model the annual salary $ X paid to people in a particular occupation where the associated probability density function is

$$f(x) = 2.5 \times 30\,000^{2.5} x^{-3.5} \text{ for } x \geq 30\,000 \text{ and 0 elsewhere.}$$

a. Find an expression in terms of a for $\Pr(X \leq a)$ where $a \geq 30\,000$. 2 marks

b. Find the mean salary of people in this occupation. 2 marks

c. Find the median salary of people in this occupation, to the nearest dollar. 3 marks

f. Find the proportion of people in this occupation who earn less than the mean salary, to the nearest percent. 2 marks

e. Find $\Pr(X > 45\,000 \mid X > 40\,000)$ correct to three decimal places. 3 marks

g. A group of 20 people in the occupation are chosen at random.
Find the probability, correct to three decimal places, that at least two of them earn more $50 000. 3 marks

Total: 15 marks

Question 6

A tank is designed to hold a particular liquid chemical. It has a capacity of 0.9 megalitres and is refilled every Monday. The weekly demand for the chemical is defined by a continuous random variable X with probability density function f, where X takes values in megalitres. The probability density function f is defined by the rule

$$f(x) = ax^2(b - x^2) \quad \text{for } x \text{ in the interval } [0, 1] \text{ and } 0 \text{ elsewhere.}$$

a. **i.** Given that the mean weekly demand is 0.625 m^3 show that $a = \dfrac{15}{6b - 4}$

3 marks

ii. State a second equation in a and b and hence show that $b = 1$ and $a = \dfrac{15}{2}$.

3 marks

b. **i.** Find $\Pr(X \le k)$ in terms of k. 2 marks

ii. Find the value of k for which $\Pr(X \le k) = \dfrac{17}{64}$ 2 marks

iii. Find, correct to two decimal places the median value of X. 2 marks

c. Find the probability that the weekly demand is greater than the capacity of the tank. 2 marks

d. Find the probability that the demand in a given week is greater than 0.8 megalitres if it is known that it is greater than 0.625 megalitres (Give your answer correct to three decimal places). 3 marks

Total: 17 marks

Question 7

Let f_a be the family of functions defined by f_a: $[0, \pi] \rightarrow R$, where

$f_a(x) = a \sin^2(x)$ and a is a positive real number.

a. Draw the graph of f_a for $a = 1$, and clearly label the coordinates of the turning point. 3 marks

b. Find the area bounded by the curve of f_1 and the x axis, and hence or otherwise find the value of a for which the corresponding function could be used to define a probability density function that assumes the value of zero for x outside the interval from 0 to π.

2 marks

c. Explain briefly why $\frac{\pi}{2}$ is the median value for a continuous random variable, X, with this probability density function.

1 mark

d. Find the mean, μ, and variance, σ^2, of X, and hence determine the interval $(\mu - 2\sigma, \mu + 2\sigma)$ with endpoints given correct to three decimal places.

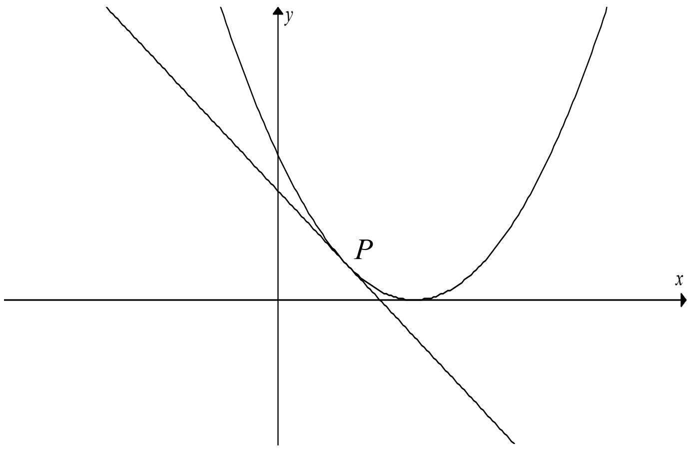
6 marks

e. Find the value of k for which $\Pr(\mu - k < X < \mu + k) = 0.95$ correct to three decimal places. 3 marks

Total: 15 marks

Question 8

The line with equation $y = mx + c$ is a tangent to the graph of the function with rule $y = (x-1)^2$ at the point P where $x = a$ and $0 < a < 1$.



a. **i.** Find the gradient of the graph of y for $x = a$ and $0 < a < 1$. 2 marks

ii. Hence express m in terms of a. 1 mark

b. State the coordinates of the point P, expressing your answer in terms of a. 2 marks

c. **i.** Show that the equation of the tangent is given by

$y = 2(a-1)x + (1-a^2)$ where $0 < a < 1$ 2 marks

ii. Find the x coordinate, in terms of a, of the point at which the tangent cuts the horizontal axis. 2 marks

d. **i.** Write down a definite integral for the area of the region enclosed by the x-axis, the tangent line and the y-axis in terms of a. 1 mark

ii. Find this area in **d.i.** in terms of a. 1 mark

e. Find the equation of the tangent for which the area enclosed by the tangent and the axes is a maximum. 3 marks

Total: 14 marks

Question 9

A mobile telephone company has a special $50 per month plan, which allows the user up to 200 minutes per month at no extra charge. Above 200 minutes, the user pays 50 cents per minute. The time used by a randomly chosen customer per month is a random variable with a normal distribution, with mean 190 minutes and standard deviation 20 minutes.

a. What proportion, correct to four decimal places, of customers exceed 200 minutes per month? 2 marks

b. If three customers are chosen at random, what is the probability that at least one will have exceeded 200 minutes in the previous month, assuming all three were on the special $50 per month plan the previous month? 3 marks

This telephone company currently has 20% of the market for this type of program. There are no long term contracts, but customers simply take out a plan for a month at a time. Of the current customers, 80% will still be customers in the next month.

c. Assume that *a*% of the rest of the market, for this type of program, switches to this telephone company from one month to the next. Write a transition matrix for this situation and find the value of *a* which is needed for the company to maintain its market share? 2 marks

The telephone company runs an advertising campaign, and hopes to pick up 10% of the rest of the market, for this type of program, each month.

c. What would its market share be one month after the advertising campaign? 2 marks

Total: 9 marks

Question 10

An architect is designing a building that has an interesting exterior shell and decides to use a cross section for the shell corresponding to the shape formed between the two curves and their points of intersection, *A* and *C*, as shown in the diagram below:

The base of the shell is modelled by the curve with equation $g(x) = \dfrac{4}{20-x}$ and the top of the shell is modelled by a curve with equation $f(x) = bx - ax^2$.

a. The difference of the functions $f(x) - g(x)$ can be written in the form $\dfrac{h(x)}{x-20}$, where $h(x)$ is a cubic polynomial. Show that $h(x) = -ax^3 + (20a + b)x^2 - 20bx + 4$.

2 marks

b. **i.** For the polynomial $h(x)$, $h(\frac{1}{4}) = 0$ and $h(\frac{39}{2}) = 0$. Use these results to form simultaneous equations in *a* and *b*. 2 marks

ii. Find the values of *a* and *b*. 2 marks

c. Find the third solution of the equation $h(x) = 0$. 2 marks

d. Find the coordinates of the points of intersection *A* and *C*, of the curves $y = f(x)$ and $y = g(x)$. 2 marks

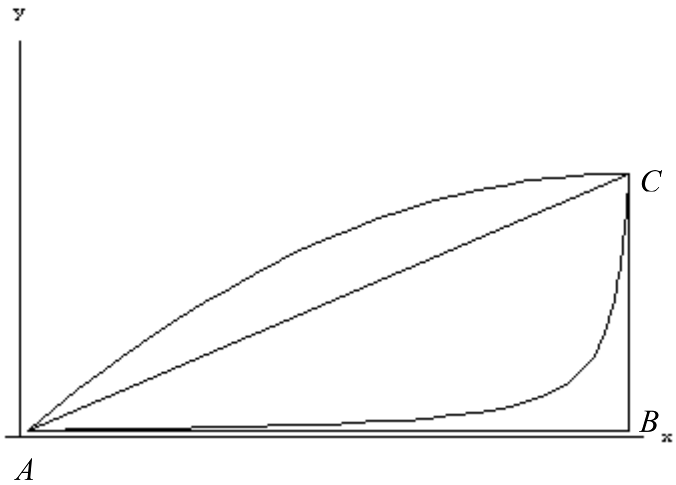
Question 10 continued next page

e. **i.** Write a definite integral which will give the exact value of the area of the cross section of the building. 2 marks

ii. Find the exact value for the area of this cross section. 1 mark

iii. Find area of the cross section correct to two decimal places. 1 mark

iv. Consider the triangle *ABC* shown in the diagram below, where B is the point of intersection of the horizontal line passing through *A* with the vertical line passing through *C*:



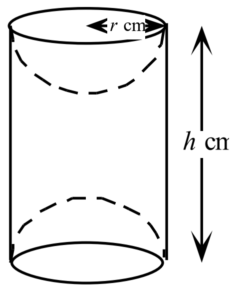
Find the area of triangle *ABC*, correct to two decimal places, and compare this area with that of your answer to **e.iii.** 2 marks

Total: 16 marks

Question 11

A container is made from plastic in the shape of a solid cylinder from which a hemisphere has been removed from each end as shown in the diagram. The total surface area of the container is 1000π cm^2.

The height of the cylinder is h cm and the radius r cm.



a. **i.** Show that $2\pi rh + 4\pi r^2 = 1000\pi$

1 mark

ii. Find h in terms of r. 2 marks

b. **i.** Find the volume of the container, V cm^3, in terms of r and h. 1 mark

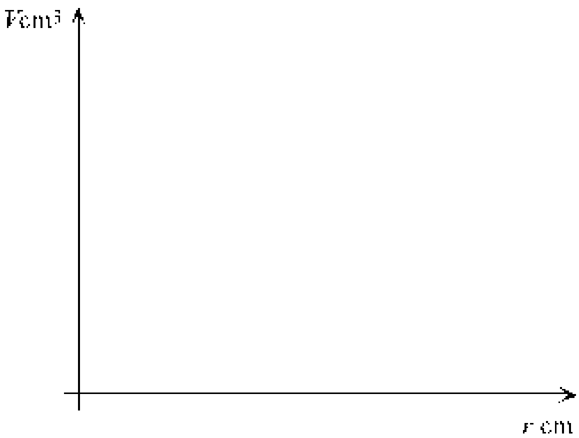
ii. Show that the volume, V cm^3, is given by $V = \frac{10\pi r}{3}(150 - r^2)$. 2 marks

c. **i.** Find $\frac{dV}{dr}$ in terms of r 1 mark

ii. Solve the equation $\frac{dV}{dr} = 0$ for r 1 mark

iii. State the maximum volume of the solid and the value of r for which this occurs. 2 marks

d. Sketch the graph of V against r on the axes provided for a suitable domain.



3 marks

Total: 13 marks

Question 12

A manufacturer produces a type of device that is a component of an electronic product. The selling price, Q,, in dollars, of each device is given by the function with rule $Q(z) = 400 - 2z$ where z is the number of devices produced. The cost, C, in dollars, of producing z devices is given by the function with rule $C(z) = 0.2z^2 + 4z + 400$.

a. **i.** Find the rule for the function R that gives the revenue in dollars from producing z devices. 1 mark

ii. Show that the profit, $\$P$, from producing and selling z devices, is given by the function with rule $P(z) = -2.2z^2 + 396z - 400$. 2 marks

b. **i.** Find the value of z for which the profit is maximised. 2 marks

ii. Find the selling price per device if this maximum profit is obtained. 2 marks

iii. Find the maximum profit. 1 mark

c. Find the possible values of z for which the profit is positive. 3 marks

e. The government imposes a tax of $22 per device. What is the selling price per device for maximum profit ? 2 marks

The same manufacturer also produces a different device. The supply and demand functions, S and D for this device, are defined by the rules $S(x) = \frac{x+1}{40} + 10$ and $D(x) = \frac{8000}{x+1}$ respectively, where x is the number of devices manufactured per week and S and D are the price in dollars per device.

e. **i.** Find the particular value x_0, such that $S(x_0) = D(x_0)$, that is, the value of x for which consumers will purchase the same quantity of devices that the manufacturer wishes to sell at that price. 2 marks

ii. State the corresponding price, p_0, for which this occurs. 1 mark

f. The consumer surplus is defined by the expression

$$\int_0^{x_0} D(x)dx - p_0 x_0 \quad .$$

Find the value of the consumer surplus to the nearest dollar. 3 marks

Total: 19 marks