THE HEFFERNAN GROUP

P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 9836 5021
Fax 9836 5025

MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 2001 SOLUTIONS

1. C	8. E	15. D	22. C
2. C	9. D	16. D	23. B
3. B	10. A	17. B	24. E
4. D	11. C	18. D	25. D
5. B	12. D	19. A	26. C
6. A	13. B	20. A	27. B
7. C	14. E	21. B	

Part I – Multiple-choice solutions

Question 1

For the function $y = \frac{A}{x-b} + B,$ where *A, b* and *B* are constants, the graph will have a vertical asymptote when $x - b = 0,$ that is, when $x = b.$ So we require that $b = -1$ so $x + 1$ will be our denominator. The horizontal asymptote is given by $y = B$ so in our case $B = 2$.

The required equation could be $y = \frac{1}{x+1} + 2.$

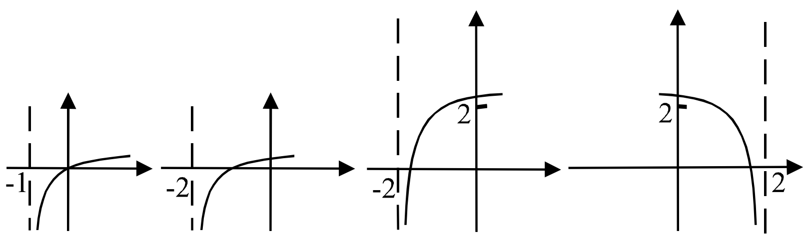
The answer is C.

Question 2

The function *f* intersects with the *x*-axis at
$x = -b$ and $x = c$ and so $x + b$ and $x - c$ are factors of f.
The function *f* touches the *x*-axis at $x = -a$ and so $x + a$ is a repeated factor of *f*. So the rule for *f* would be $y = (x+a)^2(x+b)(x-c)$.
The answer is C.

Question 3

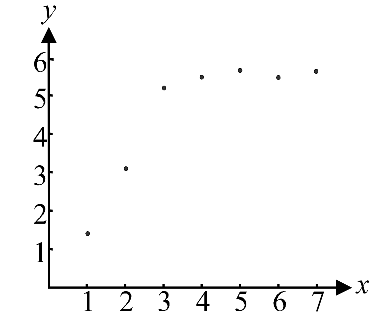
Sketch the graph of $y = \log_e(x+1)$



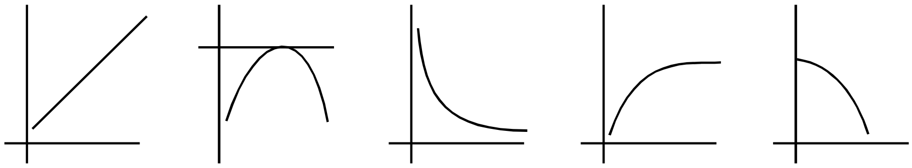
The original graph. Translated 1 unit left. Translated 2 units up. Refected in the *y* axis.

The answer is B.

Question 4
Use your graphics calculator to plot the data on a scatterplot.



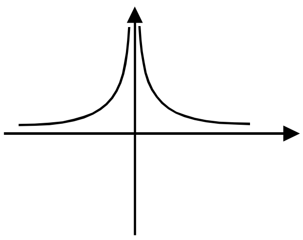
The functions given have the general shapes below:



$y = ax$ $\quad$ $y = -(x-a)^2$ $\quad$ $y = \dfrac{1}{ax}$ $\quad$ $y = \log_e ax$ $\quad$ $y = \cos(ax)$

The logarithmic function best models the data.
The answer is D.

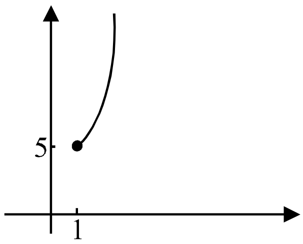
Question 5
Sketch the graph of *g*.



The domain is $(-\infty, 0) \cup (0, \infty)$.
The range is $(0, \infty)$.
The answer is B.

Question 6
Using addition of ordinates, choose some key points, for example the y – intercepts. When $x = 0$, $f(x) = 1$, $g(x) = 5$ and $h(x) = -4$.
So $f(x) = g(x) + h(x)$.
$h(x) \neq g(x) + f(x)$ and so option B is eliminated.
$g(x) \neq f(x) + h(x)$ and so option C is eliminated.
$g(x) \neq 2f(x)$ and so option D is eliminated.
$h(x) \neq -(g(x+2))$ since $h(0) = -4$ and $-g(2) = -2$ and so option E is eliminated.
Choose some other key points for example when $x = 1$ and check that the corresponding value of the functions ie. $f(1)$, $g(1)$ and $h(1)$ satisfy the rule $f(x) = g(x) + h(x)$.
The answer is A.

Question 7
Sketch $g(x)$.



From the sketch, we have $d_g = [1,\infty)$, $r_g = [5,\infty)$ so $d_{g-1} = [5,\infty)$, $r_{g-1} = [1,\infty)$

Now let $y = 3e^{x-1} + 2$

Swapping x and y gives us $x = 3e^{y-1} + 2$

Rearranging, $\frac{x-2}{3} = e^{y-1}$

So, $\log_e\left(\frac{x-2}{3}\right) = y - 1$

$$y = 1 + \log_e\left(\frac{x-2}{3}\right)$$

So we have $g^{-1} : [5,\infty) \to R$, $\quad g^{-1}(x) = 1 + \log_e\left(\frac{x-2}{3}\right)$

The answer is C.

Question 8

Use combinations:

$$^6C_3 = \frac{6!}{3!3!} = 20$$

Or use Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The x^3 term will be $20 \times (ax)^3 \times b^3 = 20(ab)^3 x^3$. The coefficient is $20(ab)^3$

The answer is E.

Question 9

$$\frac{1}{2}\log_2 x + 2\log_2 \sqrt{x} - 4\log_2 x = -5$$

$$\log_2 x^{\frac{1}{2}} + \log_2 x - \log_2 x^4 = -5 \qquad \text{(Note that } 2\log_2 \sqrt{x} = \log_2\left(x^{\frac{1}{2}}\right)^2$$

$$\log_2 \frac{x^{\frac{1}{2}} \times x}{x^4} = -5 \qquad = \log_2 x \text{)}$$

$$\log_2 \frac{x^{\frac{3}{2}}}{x^4} = -5$$

$$\log_2 x^{-2.5} = -5$$

So, $-2.5\log_2 x = -5$

So, $\log_2 x = 2$

$$2^2 = x$$

$$x = 4$$

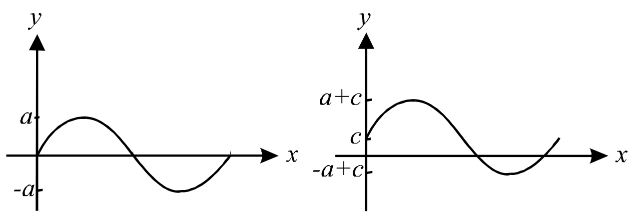
The answer is D.

Question 10

The amplitude is 1. The period is 2π and the range is $[0,2]$.
The answer is A.

Question 11

The period of *f* is 2π. The domain of *f* is $\left[-\frac{\pi}{6}, \frac{11\pi}{6}\right]$ and *f* shows one complete period of the graph of $y = a\sin(x-b)+c$. Now, *a, b* and *c* are positive constants, *a* is the amplitude, *c* is the vertical translation, and in this case *c* represents translation up $(c > 0)$.



In the first diagram, we have the minimum value of the function is $-a$ and the maximum value is a. After the function has been translated *c* units up, the minimum value is $-a+c$ (or $c-a$) and the maximum value is $a+c$.
So, the minimum value will be $c-a$ and the maximum value will be $c+a$.
The answer is C.

Question 12

The period of the graph of $y = \tan\frac{x}{2}$ is $\frac{\pi}{1/2} = 2\pi$. The asymptotes of the graph will occur at $x = \cdots -3\pi, -\pi, \pi, 3\pi, 5\pi\ldots$ The point (b,0) is halfway between the asymptotes and so could occur at $x = \cdots -2\pi, 0, 2\pi, 4\pi\ldots$ So *a* could be 3π and *b* could be 4π.
The answer is D.

Question 13

$$\sqrt{2}\cos\frac{x}{3} = -1$$

$$\cos\frac{x}{3} = \frac{-1}{\sqrt{2}}$$

$$\frac{x}{3} = \frac{3\pi}{4}, \frac{5\pi}{4}\ldots$$

$$x = \frac{9\pi}{4}, \frac{15\pi}{4}\ldots$$

Over the interval $[0,3\pi]$ there is only 1 solution and that is $\frac{9\pi}{4}$.

The answer is B.

Question 14

For $x \in (3,\infty)$ the gradient of *f* is zero so we can eliminate option C.

For $x \in (-\infty,0)$ the gradient of *f* is negative so we can eliminate A and B.

At $x = 0$ and $x = 3$ we cannot draw a tangent to the graph of *f* and so the gradient function does not exist for those values of *x*. So, we eliminate the graph of D.
The answer is E.

Question 15

$$y = \sqrt{\tan 2x}$$

$$= (\tan 2x)^{\frac{1}{2}}$$

$$\text{So, } \frac{dy}{dx} = \frac{1}{2}(\tan 2x)^{-\frac{1}{2}}.2\sec^2 2x$$

$$= \frac{\sec^2 2x}{\sqrt{\tan 2x}}$$

The answer is D.

Question 16

$$f(x) = 3x^2 \log_e(2x)$$

$$f'(x) = 6x\log_e(2x) + 3x^2 \cdot \frac{1}{x}$$

$$= 6x\log_e(2x) + 3x$$

The answer is D.

Question 17

$f(x) = e^{\cos(2x)}$

let $y = e^{\cos(2x)}$

Let $y = e^u$ where $u = \cos(2x)$

$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = -2\sin(2x)$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= e^u \cdot -2\sin(2x)$

$= -2\sin(2x)e^{\cos(2x)}$

Now, $f'(x) = -2\sin(2x)e^{\cos(2x)}$

So $f'(\pi) = -2\sin(2\pi)e^{\cos 2\pi}$

$= -2 \times 0 \times e^1$

$= 0$

Alternatively, use a graphics calculator to sketch the graph of $y = e^{\cos(2x)}$. Then calculate the value of $\frac{dy}{dx}$ when $x = \pi$. The answer is 0.

The answer is B.

Question 18

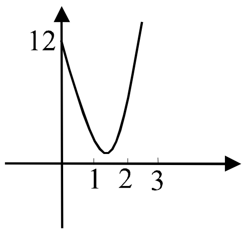
Do a quick sketch of the graph T(t) on your graphics calculator. There is a minimum turning point between t = 1 and t = 2.

Use your graphics calculator to locate this point.

It is (1.46, 1.88) to 2 places.

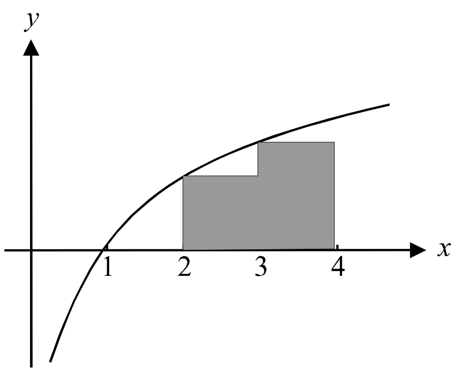
The minimum temperature is 1.9 (to 1 place).

The answer is D.



Question 19

We are looking for the shaded area shown.



Area required $= 1 \times f(1) + 1 \times f(2) + 1 \times f(3)$

$= 0 + 1 \times \log_e 2 + 1 \times \log_e 3$

$= \log_e 2 + \log_e 3$

The answer is A.

Question 20

$$\int \left(\sin(3x) + e^{-x}\right)dx = -\frac{1}{3}\cos(3x) + \frac{e^{-x}}{-1} + c$$
$$= -\frac{1}{3}\cos(3x) - e^{-x} + c$$

The answer is A.

Question 21

$$\int \frac{x^3 + 1}{\sqrt{x}}\,dx = \int \frac{x^3 + 1}{x^{\frac{1}{2}}}\,dx$$
$$= \int \left(x^{\frac{5}{2}} + x^{-\frac{1}{2}}\right)dx$$
$$= \frac{2x^{\frac{7}{2}}}{7} + 2x^{\frac{1}{2}} + c$$

An antiderivative is $\frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{1}{2}}$

The answer is B.

Question 22

The shaded area that falls below the *x*-axis will have a 'negative value'.

Hence the area required is given by $-\int_{-1}^{1} f(x)dx + \int_{1}^{4} f(x)dx - \int_{4}^{6} f(x)dx$

The answer is C.

Question 23

We have a probability distribution for a discrete random variable X and therefore

$\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) = 1$

So $\quad 2k + k + 3k + 4k = 1$

So $\quad 10k = 1$

$$k = \frac{1}{10}$$

So $\Pr(x = 1) = k$

$= 0.1$

The answer is B.

Question 24
The mean of the distribution is the x value where the bell-shaped curve peaks. For distribution X_1, this is twice that of distribution X_2.
This eliminated option A, C and D.
The variance of the distribution measures the spread of the distribution. The variance of distribution X_1 is half that of distribution X_2 and so the distribution of X_1 will not be as spread out as the distribution of X_2. This eliminates option B.
The answer is E.

Question 25
The tablets are randomly selected without replacement from the bottle and therefore we have a hypergeometric distribution.
The correct expression is D. Note that this answer could also have been written as

$$\frac{\binom{50}{5}\binom{100}{5}}{\binom{150}{10}}$$

The answer is D.

Question 26
The number of times in a week that Jack is late for his basketball commitments follows a binomial distribution where $n = 3$ and $p = 0.4$.

Now, the mean $= np$
$= 1.2$

And the variance $= np(1-p)$
$= 3 \times 0.4 \times 0.6$
$= 0.72$

The answer is C.

Question 27

The boys return what is in their net and so we have sampling with replacement. We have a binomial distribution with the probability of netting an eel being $\frac{1}{4}$.

So $\Pr(X = 2) = {}^5C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3$

The answer is B.

PART II

Question 1

a.

Number of goals (x)	0	1	2	3
$\Pr(X = x)$	$\frac{2}{20}$	$\frac{8}{20}$	$\frac{4}{20}$	$\frac{6}{20}$

(1 mark)

b. $E(x) = 0 \times \frac{2}{20} + 1 \times \frac{8}{20} + 2 \times \frac{4}{20} + 3 \times \frac{6}{20}$

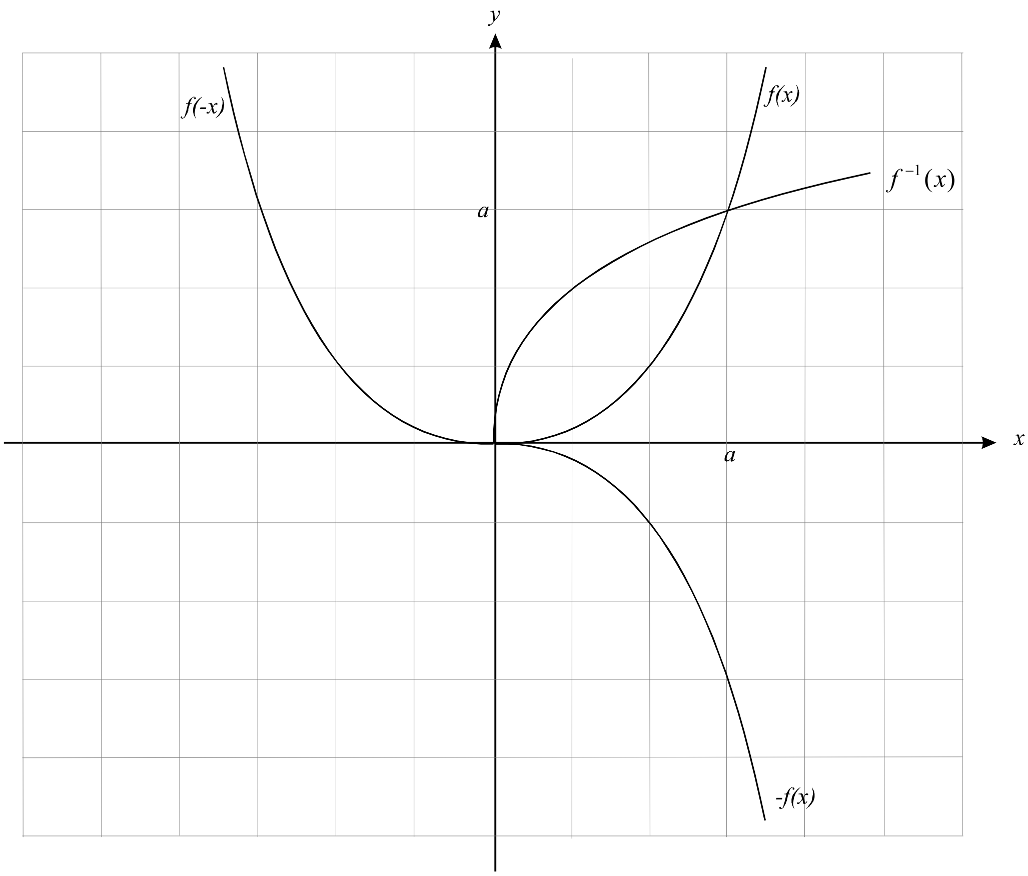
$= \frac{8}{20} + \frac{8}{20} + \frac{18}{20}$

$= \frac{34}{20}$

$= 1.7$

(1 mark)

Question 2



(3 marks)

Question 3

i. The amplitude is 2, the period is π, there is no horizontal translation and there has been a vertical translation of 1 unit up. The general equation

$y = A\sin(a(x+b)) + B$

becomes $y = 2\sin(2(x+0)) + 1$

So the function required is $f(x) = 2\sin(2x) + 1$

Check this by graphing it on your graphics calculator.

(1 mark)

ii. The amplitude is 2, the period is π, there has been a horizontal translation $\frac{\pi}{4}$ units to the right and a vertical translation of 1 unit up. The general equation

$y = A\cos(a(x+b)) + B$

becomes $y = 2\cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$

So the function required is $f(x) = 2\cos\left(2x - \frac{\pi}{2}\right) + 1$

(1 mark) for horizontal translation
(1 mark) for the complete rule

Question 4

a. Average rate of change $= \dfrac{D(2) - D(1)}{2 - 1}$

$= -2.91\text{ m/s}$ correct to 2 decimal places

(1 mark)

b. $D(t) = (t^2 - 5t)e^{0.1t}$

$D'(t) = (2t - 5)e^{0.1t} + (t^2 - 5t) \times 0.1e^{0.1t}$

(1 mark)

$D'(2) = -1.95$ m/s correct to 2 decimal places

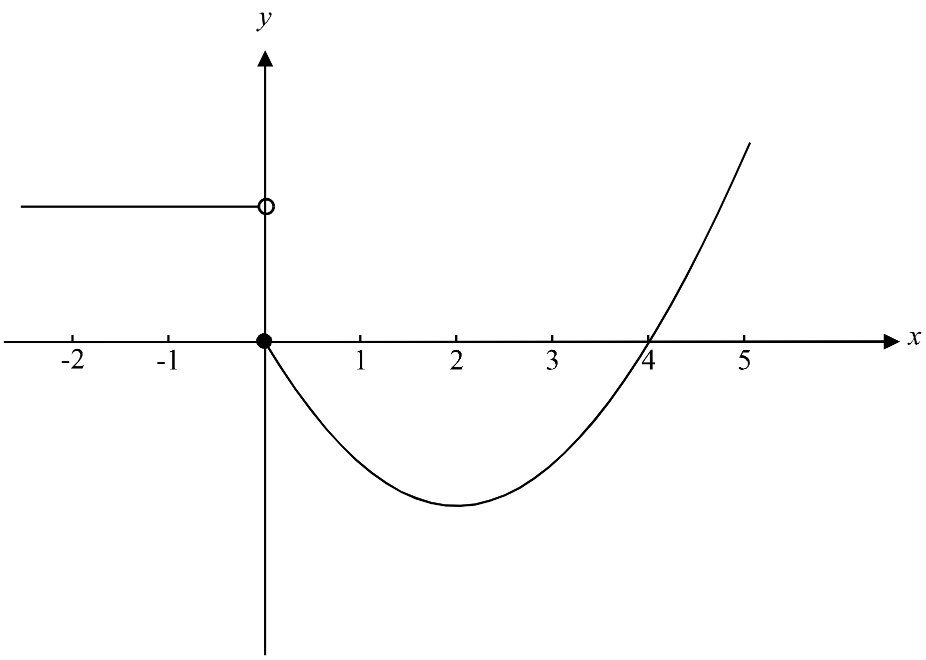
(1 mark)

Question 5

$f(0)=0$ tells us that the graph passes through the point $(0,0)$
$f(4)=0$ tells us that the graph passes through the point $(4,0)$
$f(-1)>0$ tells us that the point where $x=-1$ lies above the *x*-axis
$f'(2)=0$ tells us that at the point where $x=2$, we have a turning point or a stationary point of inflection.
$f'(x)=0, x\in(-\infty,0)$ tells us that the gradient of the curve for values of *x* less than 0 is zero and hence we have a horizontal line. Combining this with the clue $f(-1)>0$ we now know that we have a horizontal line, which runs above the *x*-axis for *x*<0.

$f'(x)<0, x\in(0,2)$ tells us that the graph has a negative gradient for values of *x* between 0 and 2.
$f'(x)>0, x\in(2,\infty)$ tells us that the graph has a positive gradient for values of *x* greater than 2.
Putting these all together we obtain the following graph.

(1 mark) for linear branch
(1 mark) for (0,0), (4,0) and turning point
(1 mark) for shape of curve



Question 6

We know that $f(x+h) \approx f(x) + hf'(x)$.

Now, $f(x) = \log_e(x^2+1)$

so, $f(2) = \log_e 5$

Also $f'(x) = \frac{2x}{x^2+1}$ **(1 mark)**

so, $f'(2) = \frac{4}{5}$

Now, $h = 2.01 - 2 = 0.01$

So $f(x+h) = f(2.01)$

$$\approx \log_e 5 + 0.01 \times \frac{4}{5}$$

$$= \log_e 5 + 0.008$$

So, $Y = \log_e 5 + 0.008$ **(1 mark)**

Question 7

a. $\Pr(1582 < X < 1642)$

$= \Pr(-1.5 < Z < 3.5)$ **(1 mark)**

$= \Pr(X < 3.5) - \Pr(X < -1.5)$

$= \Pr(X < 3.5) - (1 - \Pr(X < 1.5))$

$= 0.9998 - 1 + 0.9332$

$= 0.933$ to 3 decimal places **(1 mark)**

$Z = \frac{X-\mu}{\sigma}$

$= \frac{1642-1600}{12}$

$= \frac{42}{12}$

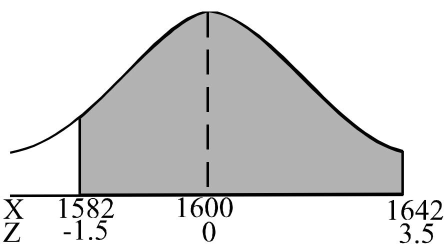
$= 3.5$

$Z = \frac{X-\mu}{\sigma}$

$= \frac{1582-1600}{12}$

$= \frac{-18}{12}$

$= -1.5$



b. Pr(X<1612) = Pr(Z<1)

= 0. 8413

$Z = \frac{X-\mu}{\sigma}$

$= \frac{1612-1600}{12}$

$= 1$

(1 mark)

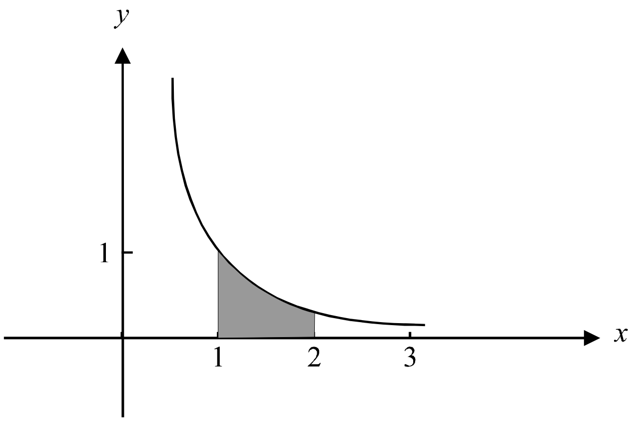
So $\Pr(x=2) = {}^3C_2 (0.8413)^2 (0.1587)^1$

$= 0.3370$ correct to 4 decimal places

(1 mark)

Question 8

Sketch a graph first.



Area required $= \int_1^2 \frac{1}{(3x-2)^{\frac{3}{2}}}\,dx$ **(1 mark)**

$$= \int_1^2 (3x-2)^{-\frac{3}{2}}\,dx$$

$$= \left[\frac{(3x-2)^{-\frac{1}{2}}}{3\times -\frac{1}{2}}\right]_1^2$$

$$= -\frac{2}{3}\left[\frac{1}{\sqrt{3x-2}}\right]_1^2$$ **(1 mark)**

$$= -\frac{2}{3}\left\{\left(\frac{1}{\sqrt{4}}\right)-\left(\frac{1}{\sqrt{1}}\right)\right\}$$

$$= -\frac{2}{3}\left(\frac{1}{2}-1\right)$$

$$= -\frac{2}{3}\times -\frac{1}{2}$$

$= \frac{1}{3}$ square units **(1 mark)**

