1998 Mathematical Methods CAT 3

Suggested Solutions

Question 1

a. $\mu + 2\sigma = 84 + 24 = 108$ mm

b. $z = \dfrac{x-\mu}{\sigma}$

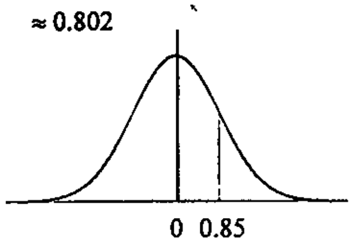
$z = \dfrac{94.2-84}{12}$

$z = 0.85$

$\Pr(x < 94.2) = \Pr(z < 0.85)$

$= 0.8023$

≈ 0.802



c. By symmetry, $\Pr(x < 75) = \Pr(x > 93)$

but the table only gives $\Pr(x < 93)$

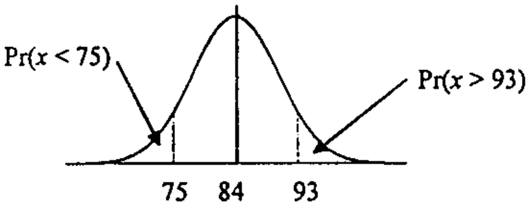
Again by symmetry, $\Pr(x > 93) = 1 - \Pr(x < 93)$

$= 1 - \Pr(z < \dfrac{93-84}{12})$

$= 1 - 0.7734$

$= 0.2266$

$= 23\%$



d. $1 - \Pr(Z < z) = 0.12$

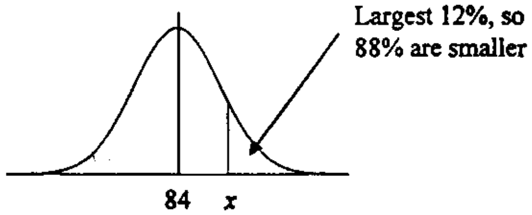
$\Pr(Z < z) = 0.88$

$z = 1.1750$

$\dfrac{x-84}{12} = 1.1750$

$x = 98.10$

$x \approx 98$ mm



e. $\Pr(X > 114 \mid X > 98) = \dfrac{[1 - \Pr(X < 114)]}{0.12}$

$= \dfrac{\left[1 - \Pr\left(z < \dfrac{114-84}{12}\right)\right]}{0.12}$

$= \dfrac{[1 - \Pr(z < 2.5)]}{0.12}$

$= \dfrac{1 - 0.9938}{0.12}$

$= 0.052$

f. $2000(0.23 \times 9 \times + 0.12 \times 30 + 0.65 \times 19) = \$360$

g. $n = 6, p = 0.12, q = 0.88, x \geq 2.$

$\Pr(X \geq 2) = 1 - [\Pr(X = 0) + \Pr(X = 1)]$

$= 1 - [0.88^6 + 6 \times 0.12 \times 0.88^5]$

$= 0.156$

Question 2

a. $R\backslash\{-2\}$ or $\{x : x \neq -2\}$ or $(-\infty, -2) \cup (-2, \infty)$

b. The graph is dilated by a factor of 12. Translated by 2 in the negative direction of the x-axis. Translated by 3 in the negative direction of the y-axis.

c. y-intercept: let $x = 0$, then $y = 6 - 3 = 3$

x-intercept: let $y = 0$

$0 = \dfrac{12}{x+2} - 3$

$3 = \dfrac{12}{x+2}$

$3x + 6 = 12$

$3x = 6$

$x = 2$

Graph of f cuts the axes at (0, 3) and (2, 0)

d. i. Let $f^{-1}(x) = y$ and let

$x = \dfrac{12}{y+2} - 3$

$x + 3 = \dfrac{12}{y+2}$

$y + 2 = \dfrac{12}{x+3}$

$y = \dfrac{12}{x+3} - 2$

$f^{-1}(x) = \dfrac{12}{x+3} - 2$

ii. All real number are allowable except when the denominator, x + 3 = 0. The domain of f^{-1} is given by $R\backslash\{-3\}$

e. Using the rule $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e(ax+b)$

$$\int_0^2 \frac{12}{x+2} - 3dx = \left[12\log_e(x+2) - 3x\right]_0^2$$
$$= 2\log_e 4 - 6 - 12\log_e 2$$
$$= 12\log_e 2 - 6$$

Question 3

a. When $x = 0, y = 0 + 0.5 \times \cos(0) = 0.5$
When $x = 0, y = 0 + 0.5 \times \cos(\pi) = 0.5$

$A = (0, 0.5), B = (\frac{\pi}{2}, 0.5)$

b. There is a stationary point at $x = \frac{\pi}{6}$ if $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \cos(x) - \sin(2x)$$
$$\text{at } x = \frac{\pi}{6}$$
$$\frac{dy}{dx} = \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$
$$= 0$$

Alternatively,

Stationary point occurs when $\frac{dy}{dx} = 0$

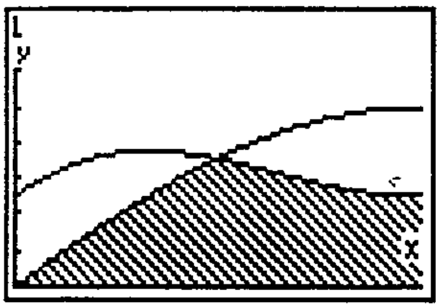
$$\cos(x) - \sin(2x) = 0$$
$$\cos(x) = \sin(2x)$$
$$\cos(x) = 2\sin(x)\cos(x)$$
$$\sin(x) = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \quad 0 \le x \le \frac{\pi}{2}$$

Hence the curve has a stationary point at $x = \frac{\pi}{6}$

c. $\sin\frac{\pi}{6} + 0.5\cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}$

$= \frac{3}{4}$

d.

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WINDOW
 Xmin=0
 Xmax=1.5708
 Xscl=.2
 Ymin=0
 Ymax=1.5
 Yscl=.2
 Xres=1
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e. let $\sin(x) + 0.5\cos(2x) = \sin(x)$

$\cos(2x) = 0$

$$2x = \frac{\pi}{2}$$
$$x = \frac{\pi}{4}$$
$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

Coordinates of intersection are $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

f. $\int_0^{\frac{\pi}{4}} \sin(x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) + 0.5\cos(2x)dx$

$$= \int_0^{\frac{\pi}{4}} \sin(x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)dx + 0.5\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)dx$$
$$= \int_0^{\frac{\pi}{2}} \sin(x)dx + 0.5\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x)dx$$
$$= \left[-\cos(x)\right]_0^{\frac{\pi}{2}} + \frac{1}{4}\left[\sin(2x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4}$$

g. $A_{north} = \int_0^{\frac{\pi}{4}} (\sin(x) + 0.5\cos(2x)) - \sin(x)dx$

$$= \int_0^{\frac{\pi}{4}} 0.5\cos(2x)dx$$
$$= \int_0^{\frac{\pi}{4}} 0.5\cos(2x)dx$$
$$= \frac{1}{2}\left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{4}(1 - 0)$$
$$= \frac{1}{4}$$

$A_{south} = 3 \times A_{north}$

Question 4

a. Total $= 10000(w\overline{XA} + \overline{AZ})$

$$\overline{XA}^2 = p^2 + 10^2$$
$$\overline{XA} = \sqrt{p^2 + 100}$$
$$\overline{AZ} = 20 - p$$
$$\text{Total} = 10000(w\sqrt{(p^2+100)} + 20 - p)$$
$$= 10000(20 - p + w\sqrt{(100+p^2)})$$
$$C = 20 - p + w\sqrt{(100+p^2)}$$

b.
$$\frac{dC}{dp} = -1 + \frac{1}{2}w(100+p^2)^{-\frac{1}{2}} \times 2p$$
$$= \frac{pw}{\sqrt{100+p^2}} - 1$$

Let $\frac{dC}{dp} = 0$

$$1 = \frac{pw}{\sqrt{100+p^2}}$$
$$\sqrt{100+p^2} = pw$$
$$p^2w^2 = 100 + p^2$$
$$p^2w^2 - p^2 = 100$$
$$p^2(w^2-1) = 100$$
$$p^2 = \frac{100}{w^2-1}$$

c. If $w = \sqrt{2}$, then $p = 10$ and the cable would start out to sea at point F of the resort. If $w > \sqrt{2}$, then p would decrease to $p < 10$ to lengthen the section of the cable following the beach. Hence it would pass partly or entirely along the beach resort.

If $w > \sqrt{2}$, $p^2 < \frac{100}{(\sqrt{2})^2 - 1}$

$$p^2 < \frac{100}{2-1}$$
$$p^2 < 100$$
$$p < 10 \text{ km}$$

d. i.
$$p^2 = \frac{100}{5-1}$$
$p = 5$ km

ii. If $w = \sqrt{5}$ and $p = 5$

Cost $= 10000(20 - 5 + \sqrt{5} \times \sqrt{100+25})$

$= 400000$

an additional \$20000 fine gives us

Total cost = \$420000

If $w = \sqrt{5}$ and $p = 10$

Cost $= 10000(20 - 10 + \sqrt{5} \times \sqrt{100+100})$

$= \$416227$

It will not be cheaper for the contractor to lay the cable if it passes in front of the resort

e. In this situation, A and Z are the same. Hence, $p = 20$

The minimum occurs when

$$p^2 = \frac{100}{w^2-1}$$

or $p = \sqrt{\frac{100}{w^2-1}}$, where $w > 1$

$$20 = \sqrt{\frac{100}{w^2-1}}$$
$$400 = \frac{100}{w^2-1}$$
$$w^2 - 1 = \frac{1}{4}$$
$$w^2 = \frac{5}{4}$$
$$w = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

The cost will be a minimum when

$$1 < w \le \frac{\sqrt{5}}{2}$$