1996
VCE
MATHEMATICAL METHODS
CAT 3

DETAILED SUGGESTED SOLUTIONS

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SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures									Letter:
Words									

Victorian Certificate of Education 1996

MATHEMATICAL METHODS

Common Assessment Task 3: Written examination (Analysis task)

Monday 11 November 1996: 9.00 am to 10.45 am
Reading time: 9.00 am to 9.15 am
Writing time: 9.15 am to 10.45 am
Total writing time: 1 hour 30 minutes

QUESTION AND ANSWER BOOKLET

Structure of booklet

Number of questions	*Number of questions to be answered*
4	4

Directions to students

Materials
Question and answer booklet of 13 pages.
Working space is provided throughout the booklet.
There is a detachable sheet of miscellaneous formulas in the centrefold.
You may bring to the CAT up to four pages (two A4 sheets) of pre-written notes.
You may use an approved calculator, ruler, protractor, set-square and aids for curve-sketching.

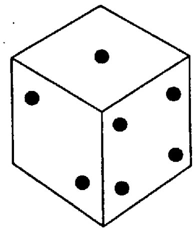
The task
Detach the formula sheet from the centre of this booklet during reading time.
Ensure that you write your **student number** in the space provided on the front cover of this booklet.
Answer **all** questions.
The marks allotted to each part of each question are indicated at the end of the part.
There is a total of 60 marks available for the task.
Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.
All written responses should be in English.

At the end of the task
You should hand in this question and answer booklet.

Question 1

Steffi has a die which is loaded, that is, it is biased. When the die is rolled, the probability of obtaining a particular number of dots (the score), X, on the uppermost face is shown in the following table.

X	1	2	3	4	5	6
$\Pr(X = x)$	p	$2p$	$3p$	$4p$	$5p$	q



a. If $p = \frac{1}{20}$, find the exact value of q.

2 marks

b. Steffi rolled the die twice.

i. Find the exact probability that both the scores were sixes.

ii. Find the exact probability that the sum of the scores was 9.

iii. Find the exact probability that, if the sum of the scores was 9, then one of the scores was 6.

1 + 3 + 3 = 7 marks

Question 1 – continued
TURN OVER

c. Steffi rolled the die 5 times. What is the exact probability of obtaining **at least 2** scores of two?

3 marks

d. Steffi rolled the die n times. Find the least value of n for which the probability of obtaining **at least 1** score of two is greater than 0.5.

4 marks

Total 16 marks

Working space

Question 2

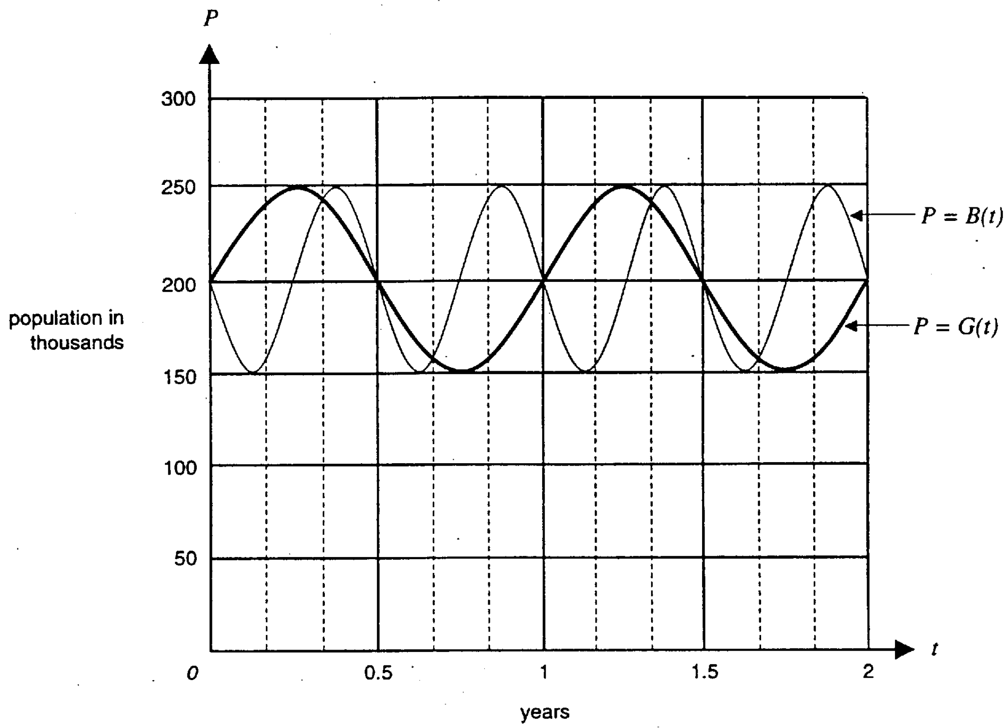
On the island of Shtam, the populations (in thousands) of two different types of insect, the Green Admiral butterfly and the Blue Captain moth, can be modelled by the following relations

Green Admiral butterfly: $G(t) = a + b\sin(mt)$

Blue Captain moth: $B(t) = c - d\sin(nt)$

where t is the time in years from when the observations of the populations started and a, b, c, d, m and n are positive constants.

The graphs of $G(t)$ and $B(t)$ for the first 2 years are shown on the axes below.



a. **i.** State the values for a, b and m and hence **write down the rule for $G(t)$.**

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ii. State the values for c, d and n and hence **write down the rule for $B(t)$.**

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2 + 2 = 4 marks

Question 2 – continued

b. How many times over a 20-year period will $B(t)$ reach its maximum value?

1 mark

c. Use the graphs to determine the times between $t = 0.25$ and $t = 0.75$ when the populations are equal, and state the values of the populations at these times.

3 marks

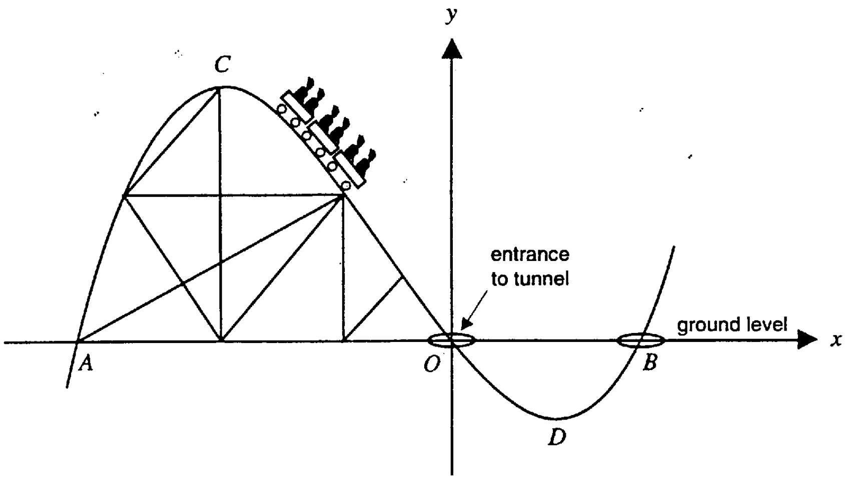
d. By solving an appropriate trigonometric equation, find after how many **days** the population of Blue Captain moths is **first** less than 180 000. Give your answer correct to ~~three decimal places~~. the nearest DAY.

3 marks

Total 11 marks

Question 3

Tasmania Jones intends to build a super roller-coaster ride, a part of the plan of which is shown on the diagram below. The ride is different from other rides in that part of the ride (from O to B) is through an underground tunnel. He has submitted his plan to the local council and he wishes to name the ride 'Tasmania's Devil'.



The track of 'Tasmania's Devil' ride follows the curve with equation

$$y = \frac{1}{720}(x^3 + 20x^2 - 1500x)$$

from A to B. C is the highest point of the ride and D is the lowest point in the tunnel. All distances are measured in metres and the equation of the curve is based on the pair of axes shown in the diagram above.

a. **i.** Factorise $x^3 + 20x^2 - 1500x$.

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ii. Hence state the coordinates of A and B.

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2 + 2 = 4 marks

Question 3 – continued

b. Show that the maximum height above the ground reached by the roller coaster is 50 metres.

3 marks

c. Find the greatest depth below the ground of the roller coaster as it passes through the tunnel from *O* to *B*. State your answer to the nearest metre.

2 marks

d. Find the average gradient of the roller-coaster track between *C* and *O*.

1 mark

Question 3 – continued
TURN OVER

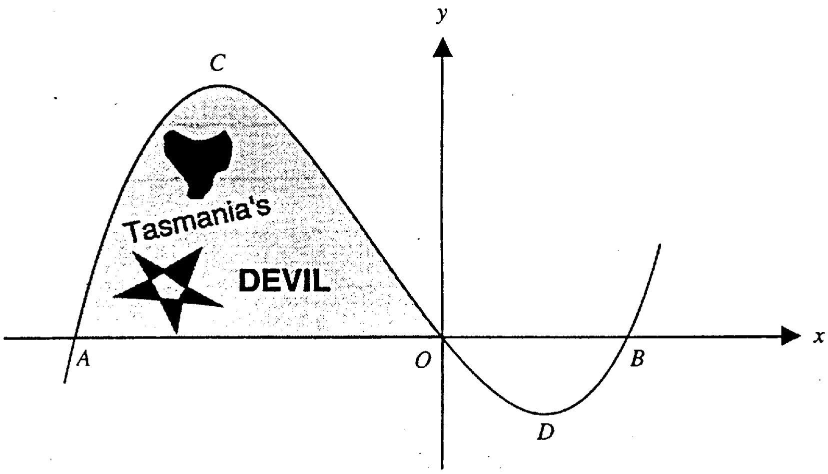
e. Find the gradient of the track as it

i. enters the tunnel at O

ii. leaves the tunnel at B.

2 marks

To advertise the ride, Tasmania Jones plans to erect a sign on the ride as shown below.



Tasmania determines the approximate area of the canvas to be used for the sign by finding the area of an appropriate triangle.

f. **i.** Draw an appropriate triangle on the diagram and use it to find an approximation to the area of the sign.

Question 3 – continued

ii. Use integration to find the area of the sign to the nearest square metre.

2 + 3 = 5 marks

Unfortunately, the council does not approve Tasmania's plan because the track is too steep at A. Tasmania has to alter the track so that the gradient of the track does not exceed 4 at any point.
He changes the shape so that the equation of the curve from A to B is

$$y = \frac{1}{N}(x^3 + 20x^2 - 1500x),$$

where N is the least number which will allow the ride to satisfy the council requirement.

g. Find N.

2 marks

h. Explain, without carrying out the calculations, how this change will affect the results of **f**.

2 marks

Total 21 marks

TURN OVER

Question 4

The current, I microamperes, passing through a particular microchip t seconds after a switch is closed, is modelled by the equation

$$I = -q(t^2 - 11t + 32)e^{t/2} + 10, \quad 0 \le t \le 6, \text{ where } q \text{ is a positive constant.}$$

a. Find the initial current in terms of q.

1 mark

b. Find an expression for $\frac{dI}{dt}$ and hence find the time t, $0 \le t \le 6$ when the current is maximum. Briefly justify that this time corresponds to a local maximum.

7 marks

Question 4 – continued

c. The maximum current in the microchip is measured to be 5 microamperes. Find the value of q correct to three decimal places.

2 marks

d. Determine whether the least value for I occurs when $t = 2$. Give reasons for your answer.

2 marks

Total 12 marks

END OF QUESTION AND ANSWER BOOKLET

1996 MATHS METHODS CAT 3

SOLUTIONS

Q1/

x	1	2	3	4	5	6
$Pr(X=x)$	p	$2p$	$3p$	$4p$	$5p$	q

a) $p = \frac{1}{20}$; Sum of probabilities is 1

$$\therefore p + 2p + 3p + 4p + 5p + q = 1$$
$$15p + q = 1$$
$$15\left(\tfrac{1}{20}\right) + q = 1$$
$$\tfrac{3}{4} + q = 1$$
$$q = \tfrac{1}{4}$$

b) i) $Pr(X=6) \times Pr(X=6)$

$= q^2$

$= \left(\frac{1}{4}\right)^2$

$= \frac{1}{16}$

ii) $Pr(\text{sum} = 9)$

$= Pr(X=3)Pr(X=6) + Pr(X=4)Pr(X=5) + Pr(X=5)Pr(X=4) + Pr(X=6)Pr(X=3)$

$= 3pq + 4p\,5p + 5p\,4p + q\,3p$

$= 6pq + 40p^2$

$= 6\left(\frac{1}{20}\right)\left(\frac{1}{4}\right) + 40\left(\frac{1}{20}\right)^2$

$= \frac{6}{80} + \frac{40}{400}$

$= \frac{70}{400}$

$= \frac{7}{40}$

(iii) $Pr(\text{one score} = 6 \mid \text{sum} = 9)$

$= \dfrac{Pr(\text{one score} = 6 \cap \text{sum} = 9)}{Pr(\text{sum} = 9)}$

$= \dfrac{Pr(X=6)Pr(X=3) + Pr(X=3)Pr(X=6)}{\frac{7}{40}} = \dfrac{6pq}{\frac{7}{40}} = \frac{6}{80} \times \frac{40}{7}$

$= \frac{3}{7}$

Q1 c) Let Y = no. of scores of two in 5 rolls of the die
Binomial, with $n = 5$, $p = 2(\frac{1}{20}) = \frac{1}{10}$

$$\begin{aligned} Pr(Y \geq 2) &= 1 - Pr(Y < 2) \\ &= 1 - [Pr(Y=0) + Pr(Y=1)] \\ &= 1 - [(\tfrac{9}{10})^5 + \binom{5}{1}(\tfrac{1}{10})(\tfrac{9}{10})^4] \\ &= 1 - [\cdot 91854] \\ &= \cdot 08146 \end{aligned}$$

d) Let Z = no. of scores of 2 in n throws of die
Binomial, where $n = n$, $p = \frac{1}{10}$

$$Pr(Z \geq 1) > 0{\cdot}5$$
$$1 - Pr(Z = 0) > 0{\cdot}5$$
$$-Pr(Z=0) > -0{\cdot}5$$
$$Pr(Z=0) < 0{\cdot}5$$
$$(\tfrac{9}{10})^n < 0{\cdot}5$$
$$\log_{10}(\tfrac{9}{10})^n < \log_{10} 0{\cdot}5$$
$$n \log_{10} {\cdot}9 < \log_{10} 0{\cdot}5$$

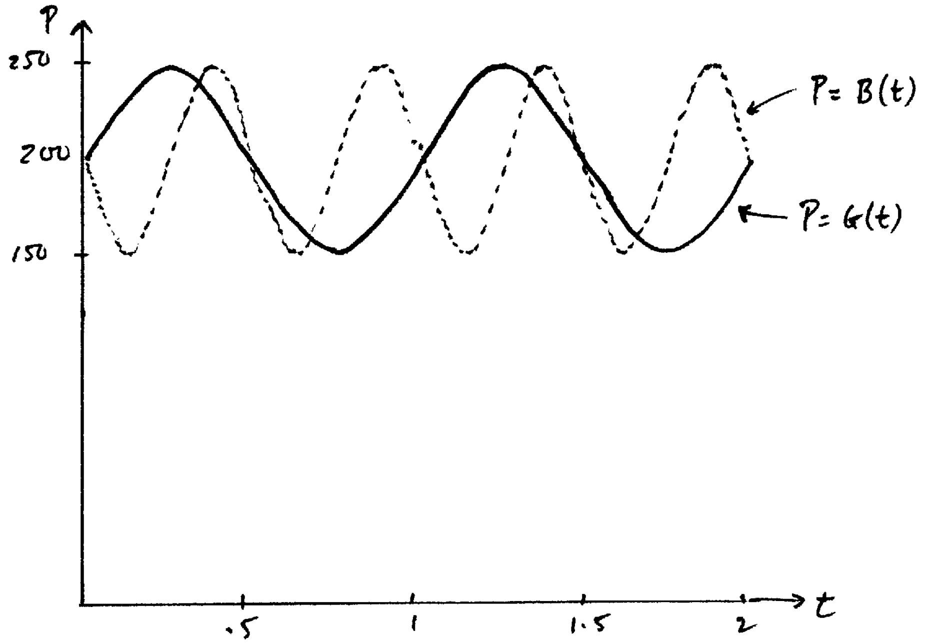
$$n > \frac{\log_{10} 0{\cdot}5}{\log_{10} 0{\cdot}9} \quad (\text{as } \log_{10} 0{\cdot}9 < 0)$$

$$n > 6{\cdot}578$$

$\therefore$ Min value of n is 7

Q2 $G(t) = a + b\sin(mt)$ t years

$B(t) = c - d\sin(nt)$ $a, b, c, d, m, n > 0$



a) i) Vertical translation of $G(t)$ is 200, so $\underline{a = 200}$

Amplitude of $G(t)$ graph is 50, so $\underline{b = 50}$

Period of $G(t)$ graph is 1, so $\frac{2\pi}{m} = 1$, so $\underline{m = 2\pi}$

$\therefore$ $G(t) = 200 + 50\sin(2\pi t)$

(ii) Vertical translation of $B(t)$ graph is 200, so $\underline{c = 200}$

Amplitude of $B(t)$ graph is 50, so $\underline{d = 50}$

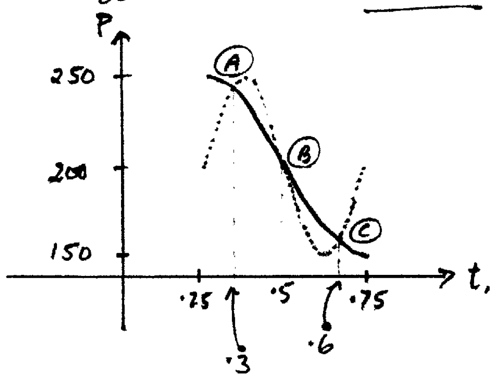
(note : − sign is due to reflection of sine graph in y-axis)

Period of $B(t)$ graph is $\cdot 5$, so $\frac{2\pi}{n} = 0\cdot5$, so $\underline{n = 4\pi}$

$\therefore$ $B(t) = 200 - 50\sin(4\pi t)$

Q2 (b) $B(t)$ reaches its maximum value twice per year (see graph)
so will do so 40 times in a 20 year period.

(c)



Populations are equal at points of intersection.
Assume vertical grid lines are equally spaced ∴ $0 \cdot 1\dot{6}$ years apart. $(\cdot 5 \div 3)$

Intersection point (A) : Population $\doteqdot$ 245 thousand at $t = \cdot\dot{3}$ years

" " (B) : Population = 200 thousand at $t = \cdot 5$ years

" " (C) : Population $\doteqdot$ 155,000 at $t = \cdot\dot{6}$ years

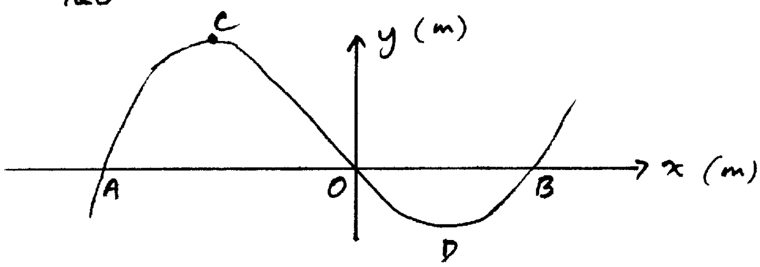
(d) Find t so that $B(t) = 180$

$$200 - 50 \sin(4\pi t) = 180$$
$$-50 \sin(4\pi t) = -20$$
$$\sin(4\pi t) = 0 \cdot 4$$
$$4\pi t = \sin^{-1}(0 \cdot 4)$$
$$= \cdot 4115 \leftarrow \text{first value (required)}$$
$$t = \frac{\cdot 4115}{4\pi} = \cdot 0327 \text{ years}$$

No. of days $= 0 \cdot 0327 \times 365$
$= 11 \cdot 95$

∴ Population is less than 180 000 after 12 days

Q3. $y = \frac{1}{720}(x^3 + 20x^2 - 1500x)$



a) (i) $x^3 + 20x^2 - 1500x$

$= x(x^2 + 20x - 1500)$

$= \underline{x(x+50)(x-30)}$

(ii) A is associated with $(x+50)$ factor, so $\underline{A(-50, 0)}$

B is associated with $(x-30)$ factor, so $\underline{B(30, 0)}$

b) Coordinates of C found by setting derivative $= 0$

$$\frac{dy}{dx} = \frac{1}{720}(3x^2 + 40x - 1500)$$

$$= \frac{1}{240}x^2 + \frac{1}{18}x - 2\tfrac{1}{12}$$

$$= \frac{1}{720}(3x - 50)(x + 30) \ *$$

so $dy/dx = 0$ when $(3x - 50) = 0$ or $(x + 30) = 0$

D → $(3x-50)$, C → $(x+30)$

For C, $x + 30 = 0 \quad \therefore \quad x = -30$

$y = \frac{1}{720}(-30)(-30+50)(-30-30)$ [from a(i)]

$= \frac{1}{720} \times -30 \times 20 \times -60$

$= \frac{36\,000}{720}$

$= 50$ $\quad \therefore$ Max height $= \underline{50\text{ m}}$

(as required)

Q3 c) from part (b), coordinates of D found from

$$3x - 50 = 0$$

$$x = \frac{50}{3}$$

$$y = \frac{1}{720}\left(\frac{50}{3}\right)\left(\frac{50}{3} + 50\right)\left(\frac{50}{3} - 30\right)$$

$$= \frac{1}{720} \times \frac{50}{3} \times \frac{200}{3} \times \frac{-40}{3}$$

$$= -\frac{400000}{19440}$$

$$= -20 \cdot 58$$

$$\sim -21$$ ∴ greatest depth is $\underline{\underline{21\,m}}$ (nearest metre)

d) Average gradient between C and O
= gradient of secant line connecting C(−30, 50) and O(0, 0)

$$m = \frac{50 - 0}{-30 - 0} = \underline{\underline{\frac{-5}{3}}}$$

e) Gradient found using derivative (see * at Q 3(b))

(i) at O, $x = 0$; ∴ $dy/dx = \frac{1}{720}(3(0) - 50)(0 + 30)$

$$= \frac{1}{720} \times -50 \times 30$$

$$= \frac{-1500}{720} = \underline{\underline{-2.08\dot{3}}}$$

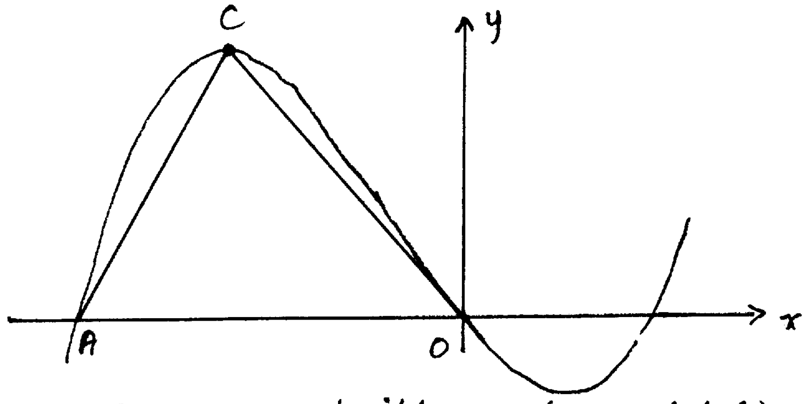
(ii) at B, $x = 30$; ∴ $dy/dx = \frac{1}{720}(3(30) - 50)(30 + 30)$

$$= \frac{1}{720}(40)(60)$$

$$= \frac{2400}{720}$$

$$= \underline{\underline{3 \cdot \dot{3}}}$$

Q3



f// (i) In $\triangle ACO$ height = 50 (y-coord of C)

base = 50

$\therefore$ Area $= \frac{1}{2}(50)(50)$

$= \underline{1250 \text{ m}^2}$

(ii) Actual area $= \int_{-50}^{0} \frac{1}{720}(x^3 + 20x^2 - 1500x)\,dx$

$= \frac{1}{720}\left[\frac{x^4}{4} + \frac{20x^3}{3} - 750x^2\right]_{-50}^{0}$

$= \frac{1}{720}\left[-\frac{(-50)^4}{4} + \frac{20(-50)^3}{3} + 750(-50)^2\right]$

$= \frac{1}{720}(1145833.\dot{3})$

$\doteq \underline{1591 \text{ m}^2}$

g// $y = \frac{1}{N}(x^3 + 20x^2 - 1500x)$

$\frac{dy}{dx} = \frac{1}{N}(3x^2 + 40x - 1500)$

$\frac{1}{N}(3x^2 + 40x - 1500) \leq 4$

$3x^2 + 40x - 1500 \leq 4N$

at $A, x = -50$: $3(-50)^2 + 40(-50) - 1500 \leq 4N$

$4000 \leq 4N$

$N \geq 1000$ ← Least value of N.

h// $\frac{1}{N}$ is a dilation factor parallel to y-axis; since N has increased, $\frac{1}{N}$ has decreased so area of triangle will decrease.

Q4. $I = -q(t^2 - 11t + 32)e^{t/2} + 10$, $0 \le t \le 6$ (seconds), $q > 0$

(a) Initial current occurs when $t = 0$

$I = -q(32)e^0 + 10$

$= (10 - 32q)$ microamperes

(b) $\frac{dI}{dt} = -q\left[(t^2 - 11t + 32)\frac{1}{2}e^{t/2} + (2t - 11)e^{t/2}\right]$ (by product rule)

$= -qe^{t/2}\left(\frac{1}{2}t^2 - \frac{11}{2}t + 16 + 2t - 11\right)$

$= -qe^{t/2}\left(\frac{1}{2}t^2 - \frac{7}{2}t + 5\right)$

$= -\frac{1}{2}qe^{t/2}(t^2 - 7t + 10)$

Maximum current occurs when $dI/dt = 0$

$-\frac{1}{2}qe^{t/2}$ cannot $= 0$, so $t^2 - 7t + 10 = 0$

$(t-2)(t-5) = 0$

$t = 2$ or 5

Test $t = 2$: at $t = 1$, $dI/dt = -\frac{1}{2}qe^{.5}(1 - 7 + 10) < 0$

at $t = 3$, $dI/dt = -\frac{1}{2}qe^{1.5}(9 - 21 + 10) > 0$

$\therefore t = 2$ is a minimum

Test $t = 5$: at $t = 3$, $dI/dt > 0$

at $t = 6$, $dI/dt = -\frac{1}{2}qe^{3}(36 - 42 + 10) < 0$

$\therefore t = 5$ is a maximum

Q4 c) $I = 5$ when $t = 5$

$$\therefore \quad 5 = -q(5^2 - 11(5) + 32)e^{5/2} + 10$$
$$-5 = -q(25 - 55 + 32)e^{5/2}$$
$$5 = q(2)e^{5/2}$$
$$q = \frac{5}{2}e^{-5/2}$$
$$\doteq .205$$

d) Test extremes of domain:

At $t = 0$, $I = 10 - 32(.205)$ (from (a))
$= 3.433$

At $t = 2$, $I = -.205(2^2 - 11(2) + 32)e^1 + 10$
$= -.205(4 - 22 + 32)e + 10$
$= -.205(14)e + 10$
≈ 2.199 ← Local min by calculus

At $t = 6$, $I = -.205(6^2 - 11(6) + 32)e^3 + 10$
$= -.205(36 - 66 + 32)e^3 + 10$
$= -.205(2)e^3 + 10$
≈ 1.765 ← Absolute minimum in interval $[0, 6]$

$\therefore$ $t = 2$ does not determine the least value for I.

END OF SUGGESTED SOLUTIONS

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