# 1996

# MATHEMATICAL METHODS

# TRIAL CAT 3

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ã **CHEMISTRY ASSOCIATES 1998**

| STUDENT NUMBER | | | | | | | | | LETTER |
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| figures | | | | | | | | | |
| words | | | | | | | | | |

# Victorian
# Mathematics 1996

# MATHEMATICAL METHODS
# 1996 TRIAL CAT 3
# Analysis Task

**Reading time: 15 minutes**
**Total writing time: 1 hour 30 minutes**

**(not to be used before Monday, October 7, 1996)**

## QUESTION AND ANSWER BOOKLET

**Directions to students**

**Materials**
Question and answer booklet of 11 pages.
Working space is provided throughout the booklet.
There is a detachable sheet of miscellaneous formulas.
You may bring to the CAT up to four pages (two A4 sheets) of pre-written notes.
You may use an approved calculator, ruler, protractor, set-square and aids for curve-sketching.

**The task**
Detach the formula sheet from this booklet during reading time.
Ensure that you write your **student number** in the space provide on the cover of this booklet.
Answer **all** questions.
The marks allotted to each part of each question are indicated at the end of the part.
There is a total of 60 marks available for the task.
Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.
All written responses should be in English.

**At the end of the task.**
Hand in this question and answer booklet.

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**Question 1**

**a.** The number of parcels in a post office each day, $X$, has the following probability distribution:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $Pr(X = x)$ | 0.10 | 0.30 | 0.20 | 0.15 | 0.15 | 0.10 |

**i.** Find the probability that a randomly selected post office box will have more than 2 parcels.

**ii.** Calculate the mean and variance for the number of parcels in a post office box.

5 marks

**b.** The Post Office decides to charge 50 cents per post box and 10 cents per parcel up to and including a total of 5 parcels
Calculate the expected charge for a post box according to the probability distribution in **part a**.

1 mark

**c.** **i.** State the name of the distribution $Z$, the number of post office boxes which contain more than 2 parcels in a sample of 20 post office boxes.

**ii.** Calculate the probability, correct to three decimal places, that from this sample of 20 post office boxes, no more than 2 post office boxes will have more than 2 parcels.

3 marks

**d.** Past records show that on average 40% of parcels in a post office box weigh more than one kilogram. Find the probability that from a randomly selected sample of 600 parcel deliveries to post office boxes, at least 245 parcels weigh more than one kilogram. Since this sample is large, use an approximation to calculate this probability correct to three decimal places.

6 marks

**Total: 15 marks**

**Question 2**

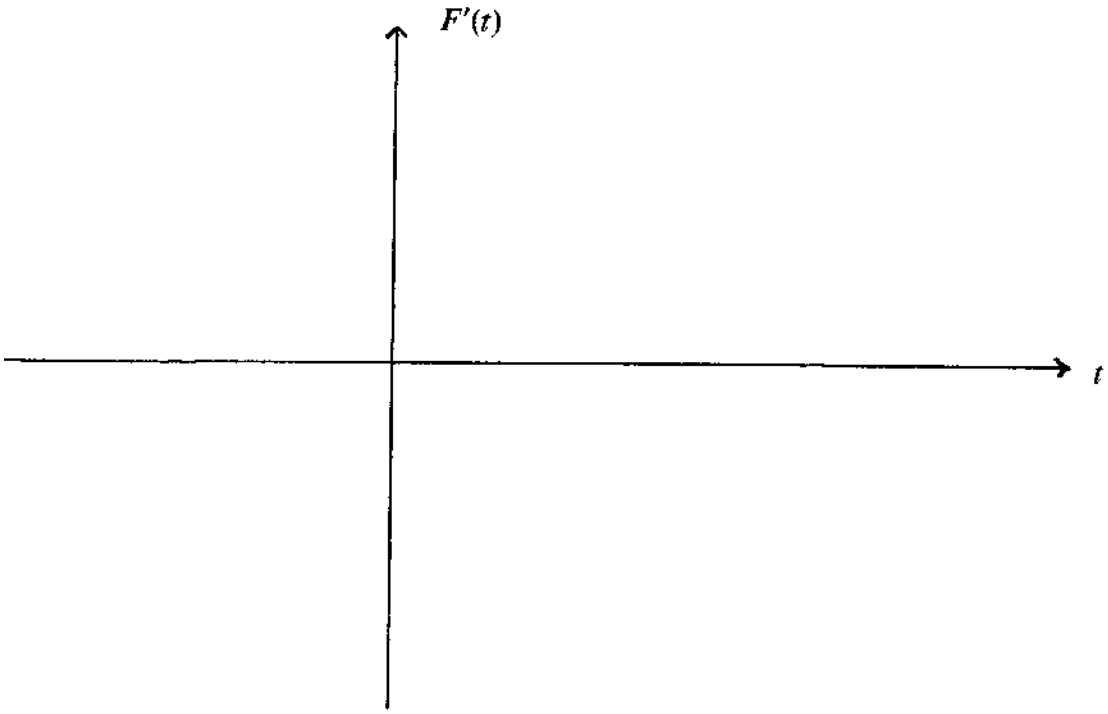
The rate at which gas burns in the heating unit in a factory is given by the function

$$F'(t) = 5 + \frac{9}{t+1}$$

where $F$ represents the amount of gas burned, in grams, $t$ minutes after the heating unit is ignited.

**a.** **i.** At what rate does the gas burn ten minutes after the heating unit is ignited?

**ii.** On the axes below, sketch the graph of $F'(t)$ against $t$.




4 marks

**b.** **i.** Find an expression for $F$, the amount of gas burned after the heating unit has been operating for $t$ minutes.

**ii.** How much gas, correct to one decimal place, is burned in the first ten minutes?

4 marks

**c.** The cost $S$ of operating a taxi while cruising at a constant speed of $v$ km/min is $S(v) = 20 + 0.04v^3$ cents per minute

**i.** Find an expression for the time taken and hence the cost of operating the taxi for a journey of 100 km at a speed of $v$ km/min.

**c.** **ii.** Find the most economical speed for this 100 km journey.

7 marks

**Total: 15 marks**

**Question 3**

The position $x$ of a particle at time $t$ is given by the equation $x = 2 \cos 2(t - \frac{\pi}{2}) + 1$.

**i** Find the position of the particle when $t = 0$.

2 marks

**ii** What is the maximum value of $x$ ?

2 marks

**iii** What is the minimum value of $x$ ?

2 marks

**Question 3 (continued)**

**iv** At what time will the particle first reach the position $x = 0$.
Give your answer to one decimal place.

5 marks

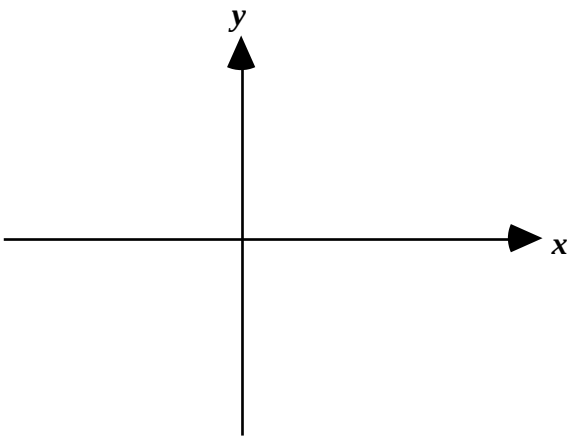
**v** Sketch the graph of $x$ for $0 \leq t \leq 2\pi$.


4 marks

**Total: 15 marks**

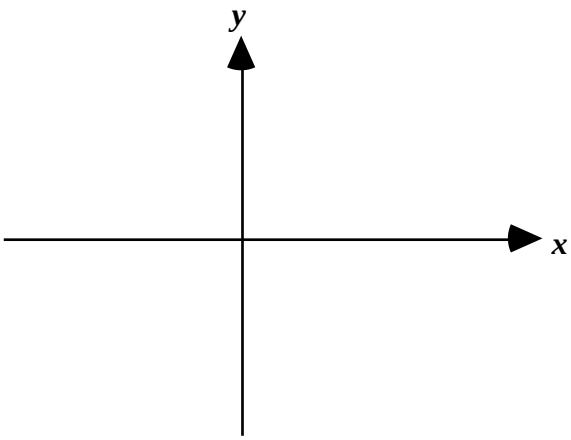
**Question 4**

**i** Sketch the graph of $y = 4 - x^2$ showing turning points and intercepts with the axes.




4 marks

**ii** A rectangle with base on the ***x*** axis has its upper vertices on the curve $y = 4 - x^2$
Show this rectangle in a sketch if the base co-ordinates are $(-a,0)$ and $(a,0)$.




2 marks

**iii** Show that the area of this rectangle equals $8a - 2a^3$.

2 marks

**Question 4 (continued)**

**iv** For what value of $a$ is the area a maximum?
Give your answer to two decimal places.

5 marks

**v** What is this maximum area? Give your answer to two decimal places.

_____

_____

_____

_____

_____

2 marks

**Total: 15 marks**

**END OF QUESTIONS 1996 MATHEMATICAL METHODS TRIAL CAT 3**

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# Suggested Solutions to 1996 Mathematical Methods Trial CAT 3

**Question 1**

**a.** **i.** $\Pr(X > 2) = \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$

$= 0.15 + 0.15 + 0.10$

$= 0.4$

**ii.**

| $x$ | $p(x)$ | $xp(x)$ | $x^2$ | $x^2p(x)$ |
|---|---|---|---|---|
| 0 | 0.10 | 0 | 0 | 0 |
| 1 | 0.30 | 0.30 | 1 | 0.30 |
| 2 | 0.20 | 0.40 | 4 | 0.80 |
| 3 | 0.15 | 0.45 | 9 | 1.35 |
| 4 | 0.15 | 0.60 | 16 | 2.40 |
| 5 | 0.10 | 0.50 | 25 | 2.50 |
| sum | | 2.25 | | 7.35 |

mean of $X = \mu = \sum xp(x) = 2.25$

$\text{var}(X) = \sum x^2 p(x) - \mu^2$

$= 7.35 - 2.25^2 = 2.29$

**b.** Let $C$ denote the charge by the Post Office.

$C = 0.1X + 0.50$

$E(C) = 0.1E(X) + 0.50$

$= 0.1 \times 2.25 + 0.50 = \$0.73$

**c.** **i.** $Z$ is Binomial

**ii.** For $Z$, $n = 20$, $p = 0.4$

$\Pr(Z \le 2) = \Pr(Z = 0) + \Pr(Z = 1) + \Pr(Z = 2)$

$= {}^{20}C_0(0.4)^0(0.6)^{20} + {}^{20}C_1(0.4)^1(0.6)^{19} + {}^{20}C_2(0.4)^2(0.6)^{18}$

$= (1)(1)(0.000036561) + (20)(0.4)(0.000060935) + (190)(0.16)(0.000101559)$

$= 0.000036 + 0.000487 + 0.003087$

$= 0.00361$

$= 0.004$

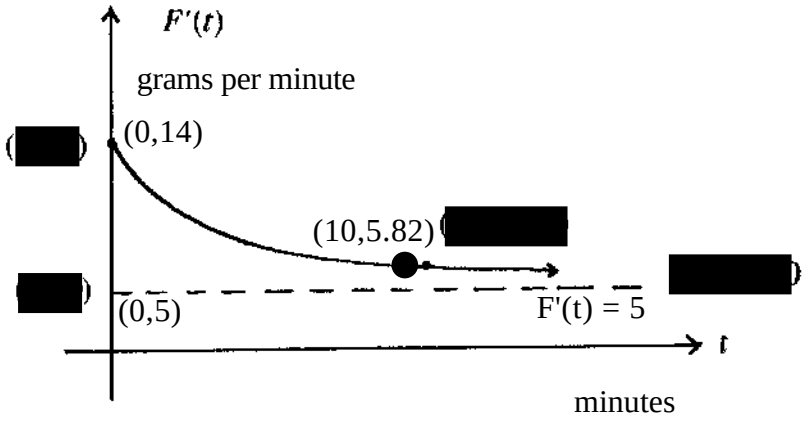
**d.** Let $X$ denote the number of parcels weighing more than one kilogram.
$X$ is Binomial with $n = 600$, $p = 0.40$
Since $n$ is large use normal approximation with $\mu = np$ and $\sigma^2 = np(1-p)$
$\mu = 600 \times 0.40 = 240$ and $\sigma^2 = 600 \times 0.40 \times 0.60 = 144$

$$\Pr(X \geq 245) = \Pr(X^* > 244.5)$$
$$= \Pr\left(Z > \frac{244.5 - 240}{\sqrt{144}}\right)$$
$$= \Pr(Z > 0.375)$$
$$= 1 - \Pr(Z < 0.375)$$
$$= 1 - 0.6462$$
$$= 0.354$$

**Question 2**

**a.** **i.**
$$F'(10) = 5 + \frac{9}{10+1} = 5.82$$
The gas is burning at a rate of 5.82 g/min after 10 minutes.

**ii.**




$$F'(t) = 5 + \frac{9}{t+1}$$
Horizontal asymptote: $F'(t) = 5$
Vertical asymptote: $t = -1$ is outside the practical domain of $t \geq 0$
$$F'(0) = 5 + \frac{9}{0+1} = 14 \text{ g/min}$$

**b.** **i.**
$$F(t) = \int \left(5 + \tfrac{9}{t+1}\right) dt$$
$$= 5t + \int \tfrac{9}{t+1}\, dt$$
$$= 5t + 9\int \tfrac{1}{t+1}\, dt$$
$$= 5t + 9\log_e(t+1) + c, \quad t \geq 0$$

**ii.** $5t + 9\log_e(t+1) + c, \quad t \geq 0$

Amount of gas burned $= F(10) - F(0)$

$$= 50 + 9\log_e 11 + c - (0 + 9\log_e 1 + c)$$
$$= 50 + 9\log_e 11$$
$$= 71.6 \text{ g}$$

**c.** **i.** $t = \dfrac{\text{distance}}{\text{speed}} = \dfrac{100}{v}$

$$S(v) = \left(20 + 0.04v^3\right)\tfrac{100}{v}$$
$$= \tfrac{20}{v} + 4v^2$$
$$= 20v^{-1} + 4v^2$$

**ii.** For local max. or min. let $S'(v) = 0$

$$-20v^{-2} + 8v = 0$$
$$\tfrac{-20}{v^2} + 8v = 0$$
$$-20 + 8v^3 = 0$$
$$8v^3 = 20$$
$$v = \sqrt[3]{2.5} = 1.36 \text{ km/min}$$

Test for minimum

$$S'(1.35) = 8(1.35) - \tfrac{20}{(1.35)^2} < 0$$
$$S'(1.37) = 8(1.37) - \tfrac{20}{(1.35)^2} > 0$$

| $v$ | 1.35 | 1.36 | 1.37 |
|---|---|---|---|
| $S'(v)$ | $< 0$ | 0 | $> 0$ |
| | \ | _ | / |

$v = 1.36$ km/min gives minimum cost and therefore is the most economical speed.

**Question 3**

(i)

When $t = 0$, $x = 2 \cos 2 (-\frac{\pi}{2}) + 1$

$= 2 \cos (-\pi) + 1$
$= 2 \cos \pi + 1$
$= 2 \times -1 + 1$
$= -2 + 1$
$= -1$ **ANS**

(ii)

The maximum value of $x$ occurs when $2 \cos 2(t - \frac{\pi}{2}) = 2$.
That is, $x = 2 + 1 = 3$ **ANS**

(iii)

The minimum value of $x$ occurs when $2 \cos 2(t - \frac{\pi}{2}) = -2$.
That is, $x = -2 + 1 = -1$ **ANS**

(iv) $2 \cos 2(t - \frac{\pi}{2}) + 1 = 0$

$2 \cos 2(t - \frac{\pi}{2}) = -1$

$\cos 2(t - \frac{\pi}{2}) = -\frac{1}{2}$

$2(t - \frac{\pi}{2}) = \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

$(t - \frac{\pi}{2}) = \frac{\pi}{3}$

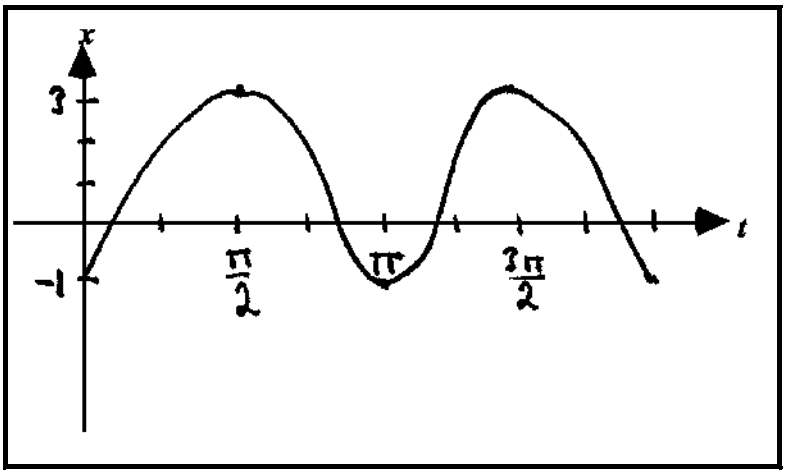
$t = \frac{\pi}{3} + \frac{\pi}{2}$

$= \frac{2\pi + 3\pi}{6}$

$= \frac{5\pi}{6}$

$t = 2.6$ **ANS**

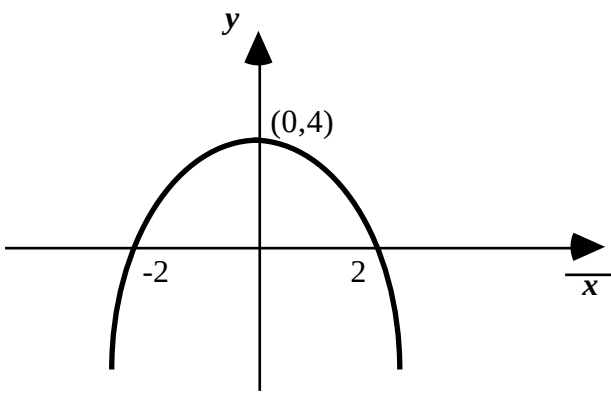
(v)



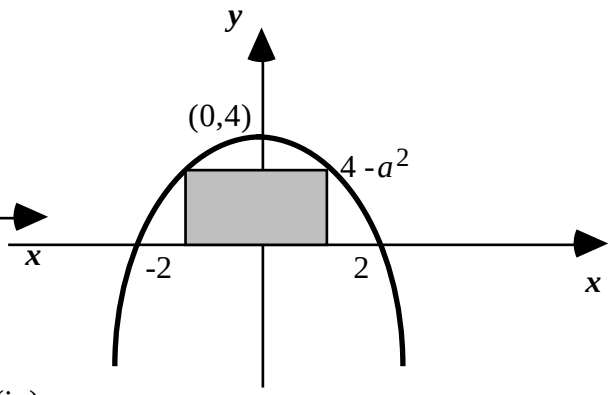
**SUGGESTED SOLUTIONS**

**Question 4**

(i)




(ii)




(iii) $A = L \times W$

$= 2a\,(4 - a^2)$

$= 8a - 2a^3$

(iv)

$\frac{dA}{dx} = 8 - 6a^2 = 0$ for a turning point.

Hence, $6a^2 = 8$

$a^2 = \frac{4}{3}$

therefore, $a = \pm \frac{2}{\sqrt{3}}$

$= \pm 1.1547$

When $a > 1.1547$ , say 2, $\frac{dA}{dx} = 8 - 24 < 0$

When $a < 1.1547$ , say 1, $\frac{dA}{dx} = 8 - 6 > 0$.

Therefore, a maximum occurs when $a = 1.15$ to two decimal places. **ANS**

(v) $A = 8a - 2a^3$

$= 8 \times \frac{2}{\sqrt{3}} - 2 \times \frac{8}{3\sqrt{3}}$

$= \frac{16}{\sqrt{3}} - \frac{16}{3\sqrt{3}}$

$= \frac{48 - 16}{3\sqrt{3}}$

$= \frac{32}{3\sqrt{3}}$

$= 6.16$ sq units. **ANS**

**END OF SUGGESTED SOLUTIONS 1996 MATHEMATICAL METHODS TRIAL CAT 3**

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