94 MM CAT 3

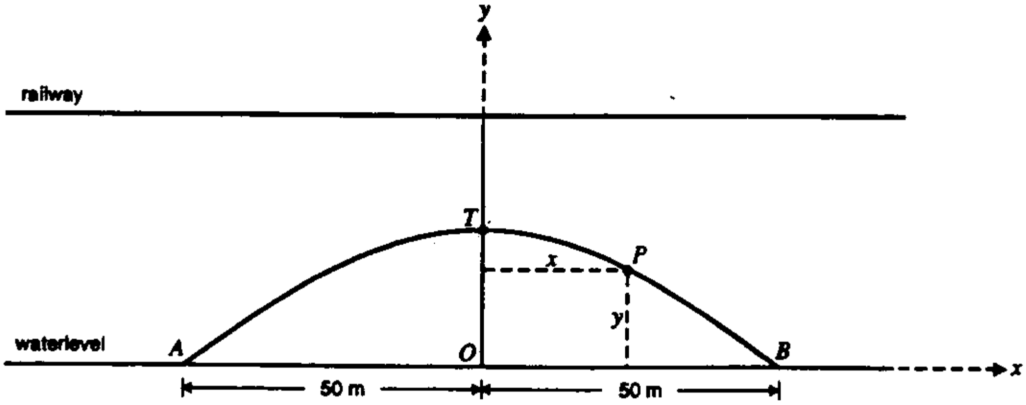
Question 1

A railway bridge over a river in countryside Victoria has a central arch. In the diagram below, the straight line AB indicates the water level under the bridge, and the curved line ATB represents the underside of the arch, and T is the top of the curve ATB.

The arch has shape given by the relation

$$y = -0.01x^2 + 25$$

where y metres is the distance from a point on the curve AB to the line AB, and x metres is the distance of the point P from the line OT.



a. What is the height of T above the water level?

25 metres

1 mark

b. The cross-sectional area enclosed by the underside of the arch and the water lies between the area of the triangle ATB and the area of the rectangle with base AB and with height OT.

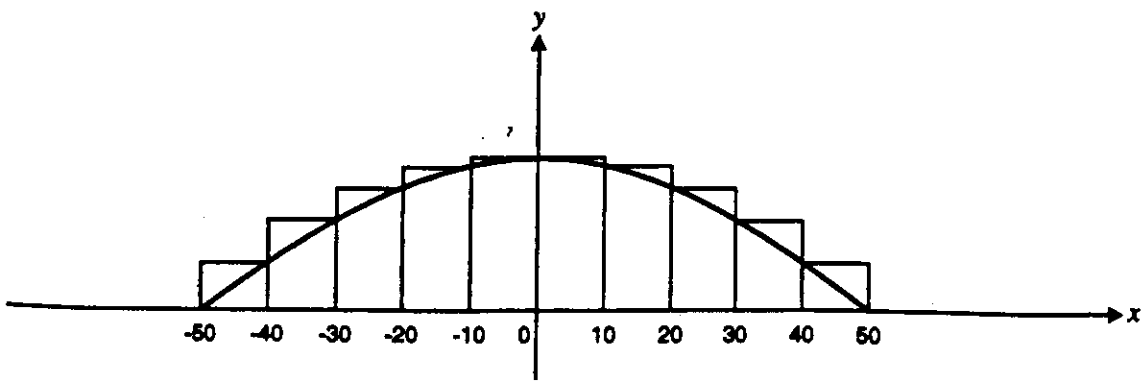
Evaluate these lower and upper approximations for the cross-sectional area.

$\triangle ATB = \frac{1}{2} \times 100 \times 25 = 1250 \text{ m}^2$

$\square ATB = 100 \times 25 = 2500 \text{ m}^2$

2 marks

c. A more accurate approximation to the cross-sectional area is made by constructing ten equally spaced 'upper rectangles', as shown on the diagram below.



Question 1 – continued

Find the value of this approximation.

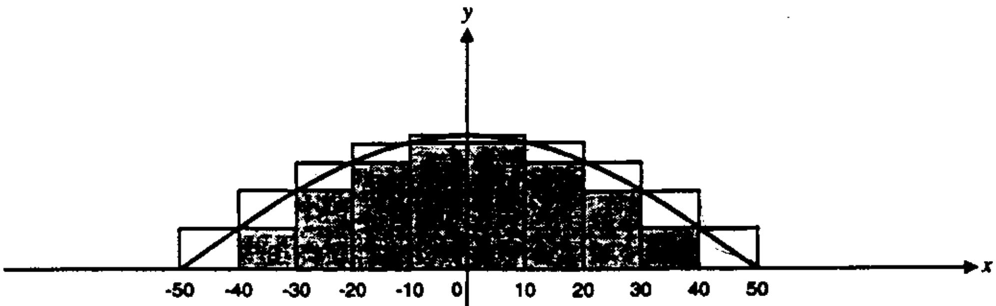
$\text{Area} \approx 2 \times 10 \times (25 + 24 + 21 + 16 + 9)$

$\quad\quad\quad\quad\quad y(0) \quad y(10) \quad y(20) \quad y(30) \quad y(40)$

$= 1900 \ m^2$

2 marks

d. Find the value of the approximation to the area obtained if the areas of the 'lower rectangles' (shaded below) are summed.



$\text{Area} \approx 2 \times 10 \times (24 + 21 + 16 + 9)$

$\quad\quad\quad\quad\quad y(10) \quad y(20) \quad y(30) \quad y(40)$

$= 1400 \ m^2$

2 marks

e. Using the information from parts **c.** and **d.**, between what two values must the actual cross-sectional area lie?

$1400 \ m^2 < \text{Area} < 1900 \ m^2$

1 mark

f. Use calculus to find the actual cross-sectional area bounded by the underside of the arch and the water level, to the nearest square metre.

$\text{Area} = 2\int_0^{50} (-0.01x^2 + 25) \, dx$

$= 2\left[-0.01\frac{x^3}{3} + 25x \right]_0^{50}$

$= 1666\frac{2}{3}$

$= 1667 \ m^2$ to nearest square metre.

2 marks

Parts **a.** and **b.** of the question were generally handled well. However, a surprising number of students were unable to cope with parts **c.** and **d.** It appeared that they had little or no understanding of the development of integral calculus and the idea of approximating areas by rectangles. Quite a number of students who tackled the rest of the paper well left these parts blank.

In part **e.** most students realised the limits were set by parts **c.** and **d.** even if they were unable to do parts **c.** and **d.** Part **f.** required students to simply use their integral calculus knowledge and many were able to do so successfully.

Again, however, many students did not give the answer to the required accuracy.

Total 10 marks

...ion 2

...mount of a radioactive substance decreases spontaneously over time because of the process of radioactive ...y. The decrease is known to be exponential, so that the amount y (gram) of a particular radioactive substance ...at after time t (years) of decay can be modelled by the equation

$$y = A\,e^{-kt} \qquad \text{(equation } \alpha\text{)}$$

...ere A and k are positive constants.

The amount y (gram) present at time t (years) in a sample of the radioactive element radium ($_{88}Ra^{226}$) is given approximately by

$$y = 1000\,e^{-0.0004\,t}$$

i. Calculate the amount of radium in the sample initially (that is, at time $t = 0$).

$y(0) = 1000\,e^{-0.0004 \times 0}$ $\qquad (0, 1000)$

$= 1000$ g

1 mark

ii. Calculate the amount of radium in the sample after 20 years.

$y(20) = 1000\,e^{-0.0004 \times 20}$

$= 992$ g

1 mark

iii. What is the average rate of decay (in grams per year) of the radium sample over the first 1000 years?

Average rate of decay $= -\left(\dfrac{y(1000) - y(0)}{1000 - 0}\right)$

$= -\left(\dfrac{670.32 - 1000}{1000}\right) = +0.33$ g/yr

2 marks

iv. Write down an equation for the instantaneous rate of change at any time t of the amount of the given sample of radium.

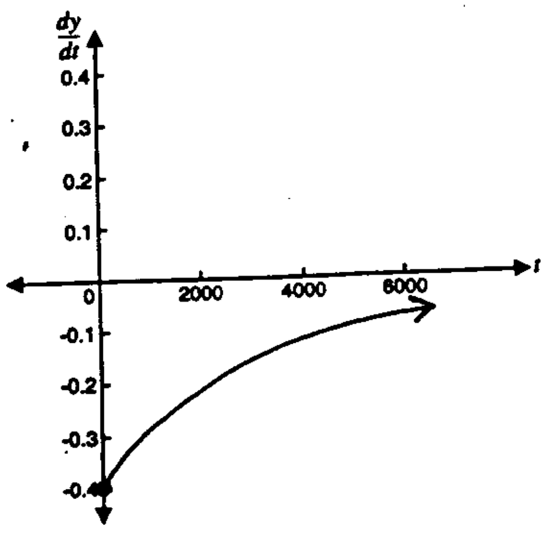
$\dfrac{dy}{dt} = 1000 \times -0.0004\,e^{-0.0004t}$

$= -0.4\,e^{-0.0004t}$

2 marks

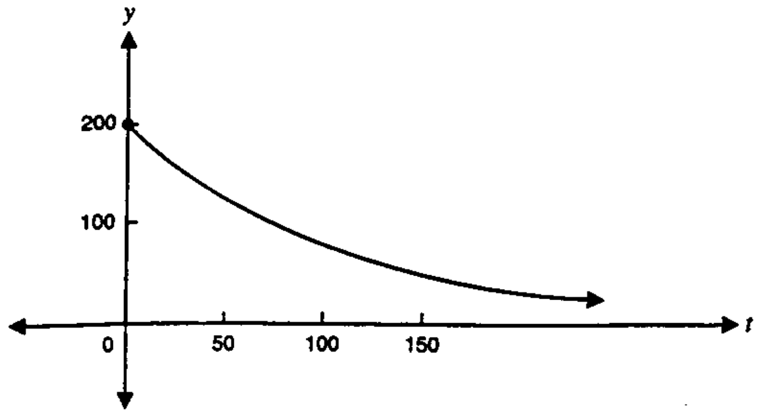
v. On the set of axes provided below, sketch the graph of the rate of change of the amount of radium versus time in the sample as it decays radioactively.

2 marks



Question 2 – continued

b. A sketch graph of the equation for the amount of a sample of the radioactive element uranium ($_{92}U^{232}$) is shown below.



It is known that the half-life of this element is 74 years. That is, it takes 74 years for a given sample of uranium to decay to half its initial amount.

The equation which models the amount of uranium (y gram) present in a given sample at any time (t years) is given by the same equation α on page 4.

From the information given above, find A and k in equation α for the element uranium. (Find k to four decimal places.)

$(74, 100)$ $\quad 100 = Ae^{-74k}$

$(0, 200)$ $\quad 200 = Ae^{0}$ $\quad \Rightarrow A = 200$

$\Rightarrow 100 = 200e^{-74k}$

$0.5 = e^{-74k}$

$-74k = \log_e 0.5$

$k = -\frac{1}{74}\log_e 0.5$

$k = 0.0094$

4 marks

Total 12 marks

Parts ai. and a.ii. were generally handled well. Once again, part a.iii. demonstrated that far too many students are unable to distinguish between instantaneous and average rate of change. Students often presented the derivative as the answer to both parts a.iii. and a.iv. In part a.iv. most students realised that the use of the chain rule for differentiation was required. However, many variations of the correct response were evident.

In part a.v. many students drew the graph well but a number had trouble recognising what was happening around the 'y-intercept' and also forgot that $t \geq 0$. Graphs were often seen in the wrong quadrant, and some students appeared to be sketching the graph of the original equation, not its derivative.

In part b. most students were able to find the value for A but were unable to correctly set up and solve the equation required to find k. Quite a few students attempted to read information from the graph to find a second equation.

• • •

Question 3

The annual rainfall of Melbourne is known to be approximately a normally distributed random variable with a mean of 660 mm and a standard deviation of 125 mm. A year is regarded as 'very wet for Melbourne' if the rainfall exceeds 820 mm. The following table gives descriptions for other rainfalls.

Rainfall for the year	Description of the year
more than 820 mm	very wet
between 595 mm and 820 mm	moderate
between 500 mm and 595 mm	fairly dry
between 485 mm and 500 mm	very dry
less than 485 mm	drought year

a. State the probability that the rainfall in 1995 will exceed 660 mm.

$Pr(X > 660) = 0.5$

1 mark

b. Calculate the probability that 1995 will have a rainfall that is 'very wet for Melbourne'. Give your answer correct to three decimal places.

$Pr(X > 820) = Pr\left(Z > \frac{820-660}{125}\right)$

$= Pr(Z > 1.28)$

$= 1 - Pr(Z < 1.28)$

$= 0.100$ (to 3 d.p.)

2 marks

c. Calculate the probability that a particular year will have a rainfall that is 'fairly dry for Melbourne'. Give your answer correct to three decimal places.

$Pr(500 < X < 595) = Pr\left(\frac{500-660}{125} < Z < \frac{595-660}{125}\right)$

$= Pr(-1.28 < Z < -0.52)$

$= Pr(Z < -0.52) - Pr(Z < -1.28)$

$= (1 - Pr(Z < 0.52)) - (1 - Pr(Z < 1.28))$

$= 0.201$ (to 3 d.p)

3 marks

d. Assuming that the weather for any year is independent of the weather for any other year, find

i. the probability that in a given three-year period, all three years will be 'fairly dry for Melbourne'. Give your answer correct to three decimal places.

Pr(3 yrs fairly dry) $= (0.201)^3$

$= 0.008$ (to 3 d.p)

2 marks

Question 3 – continued

ii. the probability that in a given seven-year period, exactly three years will be 'fairly dry for Melbourne'. Give your answer correct to three decimal places.

$Pr(3 \text{ out of } 7 \text{ yrs fairly dry}) = \binom{7}{3}(0.201)^3(0.799)^4$

$= 0.116$ (to 3 d.p)

2 marks

e. How many millimetres of rainfall annually are exceeded about 95 per cent of the time? Give your answer correct to the nearest millimetre.

let α = no. of millimetres

$Pr(X > \alpha) = 0.95$

$\Rightarrow Pr\left(Z > \frac{\alpha - 660}{125}\right) = 0.95$

$\Rightarrow Pr\left(Z < \frac{660-\alpha}{125}\right) = 0.95$

$\Rightarrow \frac{660-\alpha}{125} = 1.6449 \quad \Rightarrow \alpha = 454$ mm

4 marks

f. Show that the probability of a 'drought year' is 0.0808. Assuming the independence of the weather for one year from the weather for any other year, how often would you expect a 'drought year' to occur? Explain very briefly how you arrived at your answer.

$Pr(\text{drought}) = Pr(X < 485)$

$= Pr\left(Z < \frac{485-660}{125}\right)$

$= Pr(Z < -1.4)$

$= 1 - Pr(Z < 1.4)$

$= 1 - 0.9192$

$= 0.0808$

$\therefore$ Expect a drought every $\frac{1}{0.0808}$ years = every 12 years

(or, 8% of the time
or, 8 years in every 100 years)

4 marks

Total 18 marks

The most disappointing aspect of this question was the extremely poor notation and setting out used by a large number of students. Greater emphasis needs to be put on probability expressions and use of the standard normal curve and the reading of probability tables.

Part **a.** was generally answered well; although some students did not recognise that $Pr(x > \mu) = 0.5$.

Part **b.** was handled reasonably well, although quite a number of students left their answer as 0.1003, failing to give the answer to three decimal places (0.100) as required.

Part **c.** was handled well by a number of students. However, quite a few were unable to find the correct standard normal values and so complete the question correctly. Notation used in this part was extremely poor.

In part **d.**, quite a few students did not recognise that use of binomial was required. Many students multiplied by 3 instead of cubing.

Part **e.** produced the poorest responses in the paper. Many students incorrectly assumed that the use of two standard deviations from the mean was required. Students who performed very well overall handled this question well. It was a good discriminator and showed which students had a good understanding.

Part **f.** demonstrated that many students do not understand what is required of them when 'show that' is asked. Quite a few fumbled around to show that Pr (~~drought years~~) = 0.0808 but could not say that this meant a drought could be expected eight times every hundred years, nor why.

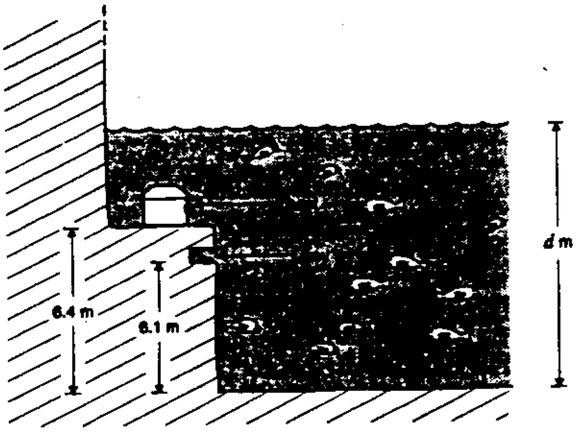
Question 4

In the depths of a jungle in Brazil, an Indian tribe keeps its treasure in a stone chest which is on a rock ledge on the banks of the piranha-infested Amazon River.

The chest cannot be moved and can only be opened by using a metal key which is kept in a hole in the rock just near the chest.

Because the piranha fish eat human flesh, the key can only be safely taken from the hole when the water level in the river falls below the level of the key.

The ledge that the chest is on is 6.4 metres above the river bed, and the hole that the key is in is 6.1 metres above the river bed.



Tasmania Jones, the intrepid adventurer, is keen to find the treasure. He is aware that the depth of water in the river could be modelled by the relation

$$d(t) = 10.0 + 4.0\cos\frac{\pi t}{14}$$

where d metres is the depth at time t hours after 12 noon on a given Monday.

a. Write down the minimum and maximum depths of the river.

min $d(14) = 10\cdot0 - 4\cdot0 = 6$ m

max $d(0) = 10\cdot0 + 4\cdot0 = 14$ m

2 marks

b. Show that the depth of water is the same at 12 noon every Monday.

period $= \frac{2\pi}{\pi/14} = 28$ or $d(0) = 14$

time from Mon to Mon $= 24 \times 7 = 168$ $d(168) = 14$

$\frac{168}{28} = 6$ complete cycles $d(336) = 14$

$\therefore$ same depth at 12 noon every Monday. $\therefore$ same depth every Monday at 12 noon

2 marks

c. Find the day and time when the water first reaches its minimum level.

$6 = 10\cdot0 + 4\cdot0\cos\frac{\pi t}{14}$

$-4 = 4\cos\frac{\pi t}{14}$

$\Rightarrow \cos\frac{\pi t}{14} = -1$ $\Rightarrow \frac{\pi t}{14} = \pi$ $\Rightarrow t = 14$

day and time is Tuesday 2 am

2 marks

Question 4 – continued

d. Find the depth of the river at

i. 12 noon on Monday ($t = 0$).

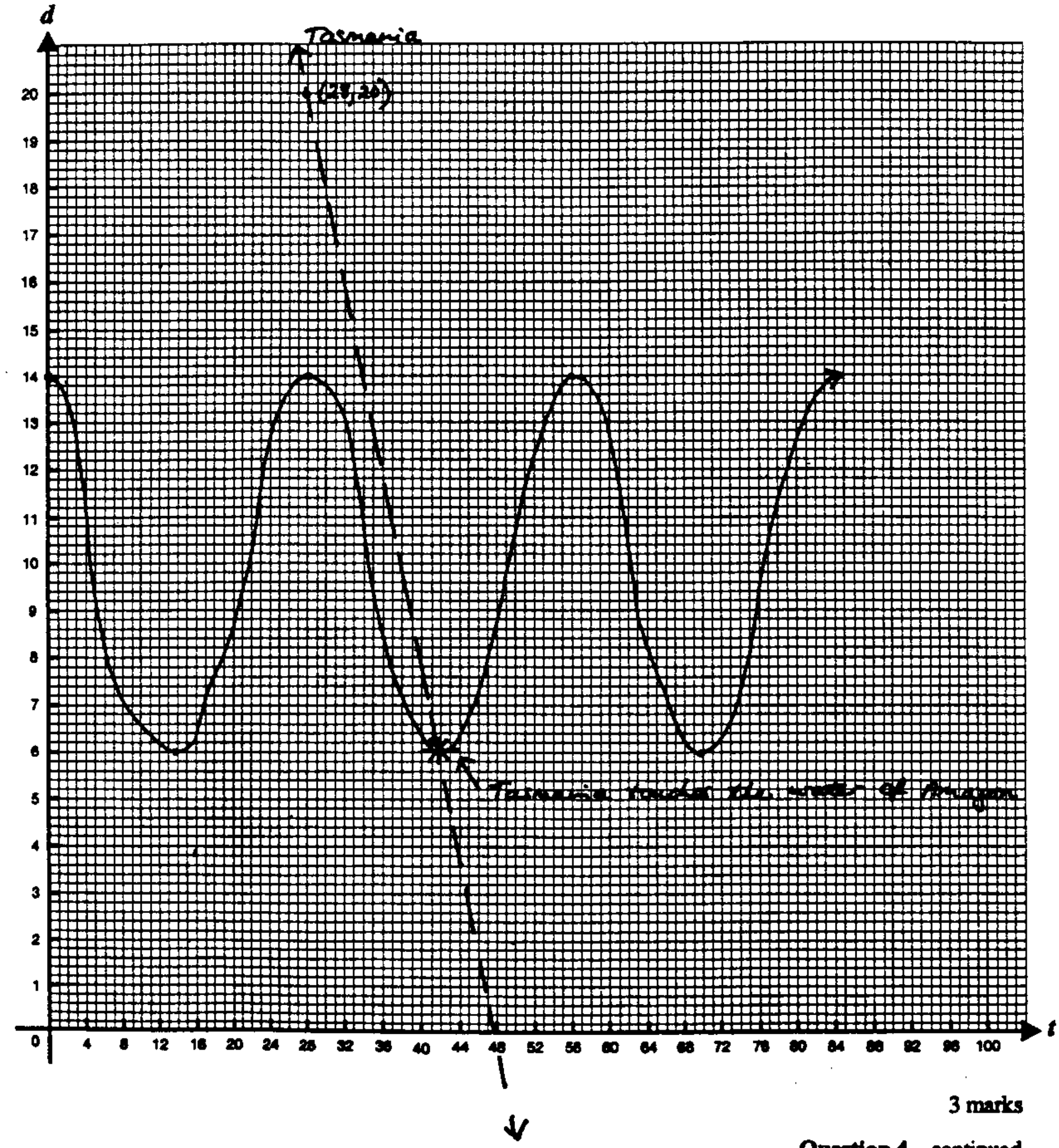
$d(0) = 10.0 + 4.0 \cos 0 = 14$ m

ii. 2 am on Tuesday ($t = 14$).

$d(14) = 10.0 + 4.0 \cos \pi = 10.0 - 4.0 = 6$ m

2 marks

e. On the set of axes provided below, sketch the graph of the depth of the river versus time showing **three** complete cycles of the graph.



3 marks

Question 4 – continued

f. Determine the first time after 12 noon on a Monday when

i. the chest is completely uncovered.

$6.4 = 10.0 + 4.0\cos\frac{\pi t}{14}$

$\Rightarrow \cos\frac{\pi t}{14} = \frac{6.4-10.0}{4.0} = -0.9$

$\Rightarrow \frac{\pi t}{14} = 2.69$

$\Rightarrow t = 12$

Midnight on Monday.

ii. the key is able to be taken from the rock.

$6.1 = 10.0 + 4.0\cos\frac{\pi t}{14}$

$\Rightarrow \cos\frac{\pi t}{14} = \frac{6.1-10.0}{4.0} = -0.975$

$\Rightarrow \frac{\pi t}{14} = 2.92$

$\Rightarrow t = 13$

1 am on Tuesday.

4 marks

g. Find out the length of time that Tasmania Jones will have to try to remove the key from the rock, take the treasure and return the key, stating your answer in minutes.

6.1 m → 6 m → 6.1 m

1 am → 2 am → 3 am

⇒ 2 hours = 120 minutes

2 marks

Unfortunately for Tasmania, the Indians capture him before he is able to obtain the treasure. They tie him by a vine 20 metres above the bottom of the river at 4 pm on a Tuesday and slowly lower him towards the water at a rate of one metre per hour.

h. Draw a graph, on the same set of axes as part e. on page 11, showing Tasmania's height above the bottom of the river versus time.

2 marks

i. Using your graph, determine the day and time at which Tasmania Jones will first touch the water of the Amazon River. Give your answer below.

$t = 42$ ⇒ 6am Wednesday.

1 mark

Total 20 marks

A surprising number of students were unable to do part **a.** but had part **d.** correct.

Part **b.** was not handled well: it was another 'show that' question. Many students simply considered two cases — insufficient to support an argument. Of those students who attempted to use the period, many stopped after finding the period equalled 28 hours.

Part **c.** was generally well done, often with little working shown. Some students gave the value for t but did not change this into the day and time as requested.

Part **d.** was generally handled well.

In part **e.**, most students had the shape of a graph correct, and either the period or the amplitude correct. Quite a number had the graph completely correct.

Parts **f.i.** and **f.ii.** demonstrated the inability of many students to solve a simple trigonometric expression. The correct equations were often set up but students were unable to solve them.

Part **g.** was not well done. Many students were unable to even demonstrate a knowledge of the cyclic nature of the problem as shown on their graph.

In part **h.**, many students had a straight line graph of about the correct gradient but not located correctly.

In part **i.**, a surprising number of students were not able to read information from their graph correctly. Some students also forgot to give the day and time, but just gave the value of t.

Chief Assessor's Comments

Generally CAT 3 demonstrated a full range of abilities: results ranging from zero to full marks were seen. The quantity of better answers was more than that seen in similar examinations in the recent past. However, it is evident that certain aspects of the course are not being given sufficient attention, such as: approximating areas by rectangles and so on; instantaneous and average rates of change; and notation — especially in probability. Student response also indicated poor use of calculators (for example, degree/radian confusion), and an inability to read questions carefully and give answers in the required form. This occurred in both CATs 2 and 3.