Neap

Trial Examination 2023

Question and Response Booklet

QCE Specialist Mathematics Units 3&4

Paper 1 – Technology-free

Student's Name: ______________________

Teacher's Name: ______________________

Time allowed

- Perusal time – 5 minutes
- Working time – 90 minutes

General instructions

- Answer all questions in this question and response booklet.
- Calculators are **not** permitted.
- Formula booklet provided.
- Planning paper will not be marked.

Section 1 (10 marks)

- 10 multiple choice questions

Section 2 (50 marks)

- 9 short response questions

SECTION 1

Instructions

- Choose the best answer for Questions 1–10.
- This section has 10 questions and is worth 10 marks.
- Use a 2B pencil to fill in the A, B, C or D answer bubble completely.
- If you change your mind or make a mistake, use an eraser to remove your response and fill in the new answer bubble completely.

	A	B	C	D
Example:	●	○	○	○

	A	B	C	D
1.	○	○	○	○
2.	○	○	○	○
3.	○	○	○	○
4.	○	○	○	○
5.	○	○	○	○
6.	○	○	○	○
7.	○	○	○	○
8.	○	○	○	○
9.	○	○	○	○
10.	○	○	○	○

SECTION 2

Instructions

- Write using black or blue pen.
- Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.
- If you need more space for a response, use the additional pages at the back of this booklet.
  - On the additional pages, write the question number you are responding to.
  - Cancel any incorrect response by ruling a single diagonal line through your work.
  - Write the page number of your alternative/additional response, i.e. See page …
  - If you do not do this, your original response will be marked.
- This section has nine questions and is worth 50 marks.

DO NOT WRITE ON THIS PAGE

THIS PAGE WILL NOT BE MARKED

QUESTION 11 (6 marks)

Consider the lines $l_1 = 4\boldsymbol{i} + 5\boldsymbol{j} - 10\boldsymbol{k} + \lambda(3\boldsymbol{i} - \boldsymbol{k})$ and $l_2 = -3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \mu(-\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$.

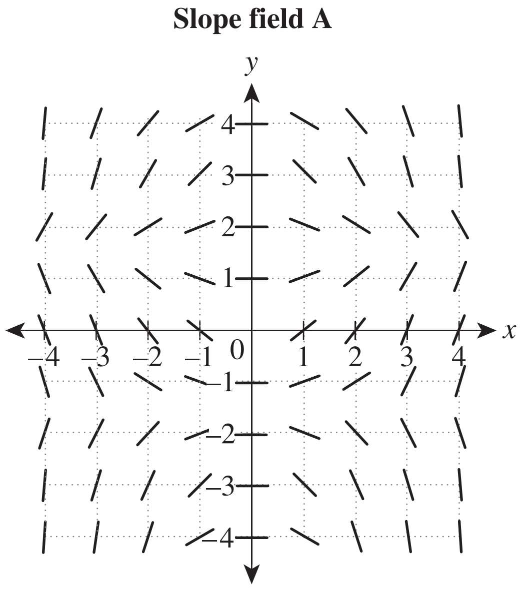
a) Determine the point of intersection of l_1 and l_2. *[2 marks]*

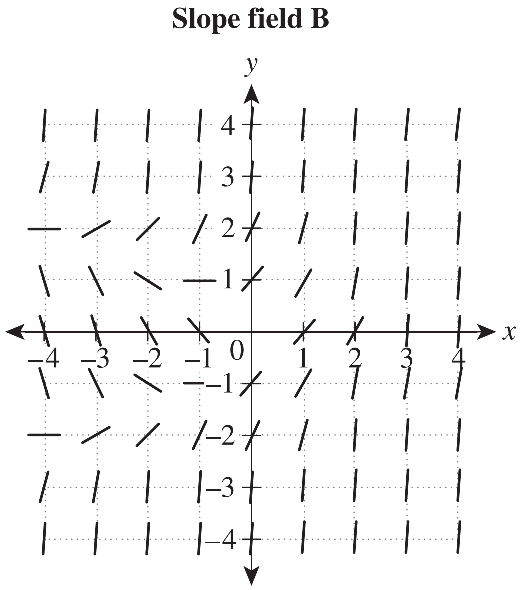
b) Show that l_1 and l_2 are perpendicular to each other. *[1 mark]*

c) Determine the equation of the vector plane that contains both l_1 and l_2 in Cartesian form. *[3 marks]*

QUESTION 12 (4 marks)

Consider the two slope field diagrams.

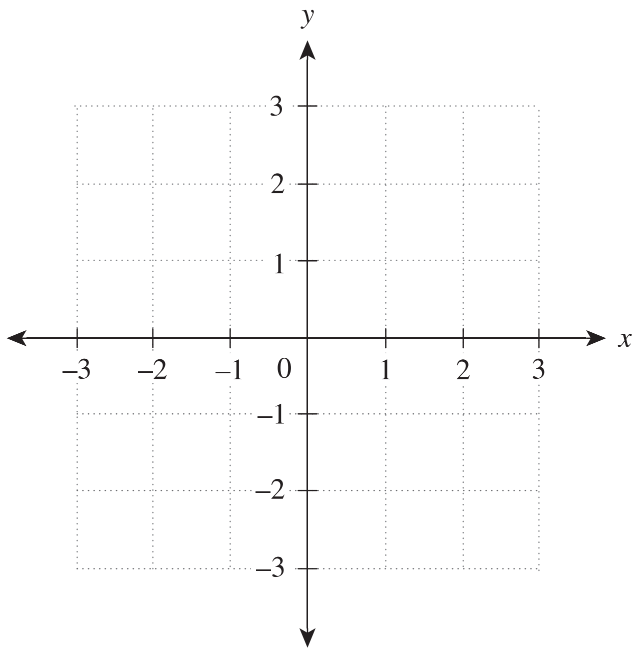




One slope field represents the differential equation $\frac{dy}{dx} = x\cos(y)$ and the other represents the differential equation $\frac{dy}{dx} = x + y^2$.

a) Identify the differential equation that is represented in slope fields A and B. Explain your reasoning. *[2 marks]*

b) Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{y}{x^2}$ on the grid. *[2 marks]*



QUESTION 13 (5 marks)

Consider the complex polynominal $p(z) = z^3 + 4z^2 + z - 26$.

a) Given that $p(2) = 0$, state one factor of $p(z)$. *[1 mark]*

b) Factorise $p(z)$ into its linear factors. *[4 marks]*

QUESTION 14 (5 marks)

Consider the system of linear equations.

$$x - 2y + z = -3$$
$$2x - 3y + 3z = -5$$
$$2y - z = 8$$

a) Write the augmented matrix for this system of linear equations. *[1 mark]*

b) Use a Gaussian technique to solve this system of linear equations. *[4 marks]*

QUESTION 15 (7 marks)

a) Using an appropriate substitution, determine the integral $\int \tan^3(x)\sec^2(x)\,dx$. *[3 marks]*

b) Determine the exact value of $\int_2^4 \frac{\ln x}{x^2}\,dx$. *[4 marks]*

QUESTION 16 (5 marks)

Prove by mathematical induction that the function $f(n) = 5^{n+1} - 4n + 11$ is divisible by 16, for all $n \in Z^{+}$.

QUESTION 17 (6 marks)

a) Solve the equation $z^3 = 4 - 4\sqrt{3}i$ using De Moivre's theorem. *[3 marks]*

b) Find the first positive integer, a, such that $z = \cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)$ is a solution to $z^a = i$. *[3 marks]*

QUESTION 18 (5 marks)

Consider the two complex numbers $z = z_1 + iz_2$ and $w = w_1 + iw_2$, where $z_1, z_2, w_1, w_2 \in R$.

Given that $|z - w| = |z + w|$, use algebraic methods to show that $\frac{w}{z}$ is purely imaginary.

QUESTION 19 (7 marks)

In many medical procedures, a glucose solution is given to a patient through an IV drip. As glucose is added to the bloodstream, it is progressively turned into other substances at a rate proportional to the concentration of glucose at that time. This occurs according to the equation $\frac{dC}{dt} = r - kC$, where:

- r is the rate at which the glucose solution is supplied
- C is the concentration of glucose in the bloodstream at time t
- k is a positive constant.

It is given that the initial concentration of glucose in a human bloodstream, C_0, is greater than 0.

Use differential equations to determine the value of C and the limit of C as time approaches infinity.

END OF PAPER

ADDITIONAL PAGE FOR STUDENT RESPONSES

Write the question number you are responding to.

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Formula Booklet

QCE Specialist Mathematics Units 3&4

Mensuration			
circumference of a circle	$C = 2\pi r$	**area of a circle**	$A = \pi r^2$
area of a parallelogram	$A = bh$	**area of a trapezium**	$A = \frac{1}{2}(a+b)h$
area of a triangle	$A = \frac{1}{2}bh$	**total surface area of a cone**	$S = \pi rs + \pi r^2$
total surface area of a cylinder	$S = 2\pi rh + 2\pi r^2$	**surface area of a sphere**	$S = 4\pi r^2$
volume of a cone	$V = \frac{1}{3}\pi r^2 h$	**volume of a cylinder**	$V = \pi r^2 h$
volume of a prism	$V = Ah$	**volume of a pyramid**	$V = \frac{1}{3}Ah$
volume of a sphere	$V = \frac{4}{3}\pi r^3$		

Calculus	
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + c$
$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \sin(x) dx = -\cos(x) + c$
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\int \cos(x) dx = \sin(x) + c$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x) dx = \tan(x) + c$
$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$	$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$
$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$	$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$

Calculus		
chain rule	If $h(x)=f\left(g(x)\right)$ then $h'(x)=f'\left(g(x)\right)g'(x)$	If $y=f(u)$ and $u=g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$
product rule	If $h(x)=f(x)g(x)$ then $h'(x)=f(x)g'(x)+f'(x)g(x)$	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
quotient rule	If $h(x)=\frac{f(x)}{g(x)}$ then $h'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
integration by parts	$\int f(x)g'(x)dx=f(x)g(x)-\int f'(x)g(x)dx$	$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$
volume of a solid of revolution	about the x-axis	$V=\pi\int_a^b\left[f(x)\right]^2dx$
	about the y-axis	$V=\pi\int_a^b\left[f(y)\right]^2dy$
Simpson's rule	$\int_a^b f(x)dx\approx\frac{w}{3}\left[f(x_0)+4\left[f(x_1)+f(x_3)+\ldots\right]+2\left[f(x_2)+f(x_4)+\ldots\right]+f(x_n)\right]$	
simple harmonic motion	If $\frac{d^2x}{dt^2}=-\omega^2x$ then $x=A\sin(\omega t+\alpha)$ or $x=A\cos(\omega t+\beta)$	
	$v^2=\omega^2\left(A^2-x^2\right)$	$T=\frac{2\pi}{\omega}$ $f=\frac{1}{T}$
acceleration	$a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$	

Real and complex numbers	
complex number forms	$z=x+yi=r\left(\cos(\theta)+i\sin(\theta)\right)=r\,\mathrm{cis}(\theta)$
modulus	$\lvert z\rvert=r=\sqrt{x^2+y^2}$
argument	$\arg(z)=\theta,\ \tan(\theta)=\frac{y}{x},-\pi<\theta\le\pi$
product	$z_1z_2=r_1r_2\,\mathrm{cis}\left(\theta_1+\theta_2\right)$
quotient	$\frac{z_1}{z_2}=\frac{r_1}{r_2}\mathrm{cis}\left(\theta_1-\theta_2\right)$
De Moivre's theorem	$z^n=r^n\mathrm{cis}(n\theta)$

Statistics		
binomial theorem	$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{r} x^{n-r} y^r + \ldots + y^n$	
permutation	${}^nP_r = \dfrac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$	
combination	${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$	
sample means	mean	μ
	standard deviation	$\dfrac{\sigma}{\sqrt{n}}$
approximate confidence interval for $\boldsymbol{\mu}$	$\left(\bar{x} - z\dfrac{s}{\sqrt{n}}, \bar{x} + z\dfrac{s}{\sqrt{n}} \right)$	

Trigonometry	
Pythagorean identities	$\sin^2(A) + \cos^2(A) = 1$ $\tan^2(A) + 1 = \sec^2(A)$ $\cot^2(A) + 1 = \operatorname{cosec}^2(A)$
angle sum and difference identities	$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$ $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
double-angle identities	$\sin(2A) = 2\sin(A)\cos(A)$ $\cos(2A) = \cos^2(A) - \sin^2(A)$ $= 1 - 2\sin^2(A)$ $= 2\cos^2(A) - 1$
product identities	$\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$ $\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$ $\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$ $\cos(A)\sin(B) = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$

Vectors and matrices		
magnitude	$\lvert \boldsymbol{a} \rvert = \left\lvert \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\rvert = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$	
scalar (dot) product	$\boldsymbol{a} \cdot \boldsymbol{b} = \lvert \boldsymbol{a} \rvert \lvert \boldsymbol{b} \rvert \cos(\theta)$	
	$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$	
vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + k\boldsymbol{d}$	
Cartesian equation of a line	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$	
vector (cross) product	$\boldsymbol{a} \times \boldsymbol{b} = \lvert \boldsymbol{a} \rvert \lvert \boldsymbol{b} \rvert \sin(\theta) \hat{\boldsymbol{n}}$	
	$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$	
vector projection	$\boldsymbol{a}$ on $\boldsymbol{b} = \lvert \boldsymbol{a} \rvert \cos(\theta) \hat{\boldsymbol{b}} = \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}} \right) \hat{\boldsymbol{b}}$	
vector equation of a plane	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$	
Cartesian equation of a plane	$ax + by + cz + d = 0$	
determinant	If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(\mathbf{A}) = ad - bc$	
multiplicative inverse matrix	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $\det(\mathbf{A}) \neq 0$	
linear transformations	dilation	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
	rotation	$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
	reflection (in the line $y = x\tan(\theta)$)	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Physical constant	
magnitude of mean acceleration due to gravity on Earth	$g = 9.8 \text{ m s}^{-2}$