Neap

Trial Examination 2022

**Suggested Solutions**

# QCE Specialist Mathematics Units 3&4

## Paper 1 – Technology-free

### SECTION 1 – MULTIPLE CHOICE QUESTIONS

| | A | B | C | D |
|---|---|---|---|---|
| **1.** | ○ | ● | ○ | ○ |
| **2.** | ○ | ● | ○ | ○ |
| **3.** | ○ | ○ | ● | ○ |
| **4.** | ○ | ● | ○ | ○ |
| **5.** | ● | ○ | ○ | ○ |
| **6.** | ○ | ○ | ○ | ● |
| **7.** | ○ | ● | ○ | ○ |
| **8.** | ○ | ● | ○ | ○ |
| **9.** | ● | ○ | ○ | ○ |
| **10.** | ○ | ● | ○ | ○ |

**QUESTION 1 B**

$$\begin{aligned}\frac{x-1}{x^2-2x-15} &= \frac{x-1}{(x-5)(x+3)}\\ &= \frac{A}{x-5}+\frac{B}{x+3}\\ &= \frac{A(x+3)+B(x-5)}{(x-5)(x+3)}\\ &= \frac{(A+B)x+3A-5B}{(x-5)(x+3)}\end{aligned}$$

This gives two equations: $A + B = 1$ (equation 1) and $3A - 5B = -1$ (equation 2). Rearranging equation 1 gives $B = 1 - A$ (equation 3).

Substituting equation 3 into equation 2 gives:

$$\begin{aligned}3A-5(1-A) &= -1\\ 3A-5+5A &= -1\\ 8A &= 4\\ A &= 0.5\end{aligned}$$

Substituting $A = 0.5$ into equation 1 gives:

$$\begin{aligned}0.5+B &= 1\\ B &= 0.5\end{aligned}$$

**QUESTION 2 B**

The equation to calculate the area of a triangle with two sides $\boldsymbol{p}$ and $\boldsymbol{q}$ is $\frac{1}{2}|\boldsymbol{p}\times\boldsymbol{q}|$. In this case, two of the sides are described by the vectors $(\boldsymbol{a} - \boldsymbol{b})$ and $(\boldsymbol{b} - \boldsymbol{c})$. Therefore, the answer is $\frac{1}{2}|(\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{b}-\boldsymbol{c})|$.

**QUESTION 3 C**

**C** is correct. The definition of circular motion can be stated as $\boldsymbol{v}\cdot\boldsymbol{r}=0$.

**A** is incorrect. This option defines linear acceleration.

**B** is incorrect. This option allows displacement to be calculated as the anti-derivative of velocity.

**D** is incorrect. This option defines vertical projectile motion under gravity.

**QUESTION 4 B**

$(2-3i)^3(5-i)$

$(8-36i+54i^2-27i^3)(5-i)$

$(-230+46i-45i+9i^2)$

$(-239+1i)$

$\text{Im}(-239+1i)=1$

 TEQSM34_SS_P1_2022 

**QUESTION 5 A**

$$u^3 = 8\text{cis}(\pi)$$
$$= -8$$
$$v^2 = -2 + 2\sqrt{3}i$$
$$\frac{u^3}{v^2} = \frac{-8}{-2+2\sqrt{3}i}$$
$$= \frac{-4}{-1+\sqrt{3}i}$$
$$= \frac{-4}{-1+\sqrt{3}i} \times \frac{-1-\sqrt{3}i}{-1-\sqrt{3}i}$$
$$= \frac{4+4\sqrt{3}i}{4}$$
$$= 1+\sqrt{3}i$$

**QUESTION 6 D**

$$\frac{dy}{dx} = \frac{x^2 y}{5}$$
$$\frac{dy}{y} = \frac{1}{5}x^2\,dx$$
$$\ln(y) = \frac{1}{15}x^3 + C$$
$$y = ke^{\frac{x^3}{15}}$$

**QUESTION 7 B**

The substitution of $u = 5x^2 + 2x$ and $\frac{du}{dx} = 10x + 2$ allows the sum to be written as:

$$\int (10x+2)e^{\left(5x^2+2x\right)}dx = \int \frac{du}{dx}e^u\,dx$$
$$= \int e^u\,dx$$

**QUESTION 8 B**

$$\mu_{\bar{x}} = \mu,\ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$
$$\mu_{\bar{x}} = \mu \Rightarrow \mu = 75 \text{ minutes}$$
$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$
$$\sigma = \sigma_{\bar{x}} \times \sqrt{n}$$
$$= 6 \times \sqrt{100}$$
$$= 6 \times 10$$
$$= 60 \text{ minutes}$$

**QUESTION 9 A**

By inspection, $8x$ is the derivative of $4x^2 + 8$ (the expression in the brackets). Alternatively, a substitution of $u = 4x^2 + 8$ and $\frac{du}{dx} = 8x$ can be used to simplify the integration.

$$\begin{aligned}\int 8x\left(4x^2+8\right)^4 dx &= \int \frac{du}{dx}(u)^4 dx \\ &= \int (u)^4 du \\ &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}\left(4x^2+8\right)^5 + C\end{aligned}$$

**QUESTION 10 B**

**B** is correct. Increasing the sample size will reduce the sample standard deviation as the mean will be more certain.

**A** and **D** are incorrect. Increasing the sample size will not change the sample or population mean of the object being sampled.

**C** is incorrect. If the sample size is increased by a factor of 5, a 95% confidence interval will reduce by a factor of $\sqrt{5}$.

## SECTION 2

### QUESTION 11 (5 marks)

a) $\overrightarrow{AB} = (3 - -2, 5 - 5, -2 - 8)$

$= (5, 0, -10)$ or $5\hat{\boldsymbol{i}} - 10\hat{\boldsymbol{k}}$

$\overrightarrow{AC} = (4 - -2, -1 - 5, 1 - 8)$

$= (6, -6, -7)$ or $6\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$

*[2 marks]*
*1 mark for determining $\overrightarrow{AB}$.*
*1 mark for determining $\overrightarrow{AC}$.*

b) $\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{a}} + k\tilde{\boldsymbol{d}}$

$= (-2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}}) + k(6\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}})$

*[1 mark]*
*1 mark for identifying a suitable representation of $\tilde{\boldsymbol{r}}$.*
*Note: Consequential on answer to **Question 11a)**.*

c) The radius, $r$, is equal to the distance between points A and B.

$r = \sqrt{5^2 + 0^2 + 10^2}$

$r = \sqrt{125}$

$r^2 = 125$

Cartesian equation of the sphere centred on B(3, 5, –2) with a radius of $\sqrt{125}$:

$125 = (x - 3)^2 + (y - 5)^2 + (z + 2)^2$

*[2 marks]*
*1 mark for calculating the radius.*
*1 mark for identifying the Cartesian equation of the sphere.*

**QUESTION 12 (4 marks)**

$$\int_0^{\pi}\cos\left(\frac{3x}{4}\right)\cos\left(\frac{x}{4}\right)dx=\frac{1}{2}\int_0^{\pi}\left(\cos\left(\frac{3x}{4}+\frac{x}{4}\right)+\cos\left(\frac{3x}{4}-\frac{x}{4}\right)\right)dx$$

$$=\frac{1}{2}\int_0^{\pi}\left(\cos(x)+\cos\left(\frac{x}{2}\right)\right)dx$$

$$=\frac{1}{2}\left[\sin(x)+2\sin\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$=\frac{1}{2}\left(\sin(\pi)+2\sin\left(\frac{\pi}{2}\right)-\sin(0)-2\sin\left(\frac{0}{2}\right)\right)$$

$$=\frac{1}{2}(0+2-0-0)$$

$$=1$$

*[4 marks]*

*1 mark for converting* $\cos\left(\frac{3x}{4}\right)\cos\left(\frac{x}{4}\right)$ *to* $\left(\cos(x)+\cos\left(\frac{x}{2}\right)\right)$.

*1 mark for integrating to* $\frac{1}{2}\left[\sin(x)+2\sin\left(\frac{x}{2}\right)\right]_0^{\pi}$.

*1 mark for expanding the integral. Note: This may be implied by consequent working.*

*1 mark for providing the correct solution.*

**QUESTION 13 (5 marks)**

a) $a = 9 - 0.2v$

To find the terminal velocity, $v$ must be found at $a = 0$.

$0 = 9 - 0.2v$

$v = \frac{9}{0.2}$

$= 45 \text{ m s}^{-1}$

*[2 marks]*

*1 mark for identifying the formula for acceleration.*

*1 mark for calculating the terminal velocity.*

b) $$\frac{dv}{dt} = 9 - 0.2v$$

$$\frac{dv}{9 - 0.2v} = dt$$

$$\frac{\ln(9 - 0.2v)}{-0.2} = t + C$$

$$9 - 0.2v = e^{-0.2C} e^{-0.2t}$$

At $t = 0$, $v = 0$. Therefore, $e^{-0.2C} = 9$.

$$0.2v = 9 - 9e^{-0.2t}$$

$$v = 45\left(1 - e^{-0.2t}\right)$$

*[3 marks]*
*1 mark for identifying the differential equation.*
*1 mark for integrating the differential equation.*
*1 mark for solving the differential equation.*

**QUESTION 14 (7 marks)**

Let $u = 5x^2$ and $\frac{dv}{dt} = e^{0.5x}$.

Therefore, $\frac{du}{dt} = 10x$ and $v = 2e^{0.5x}$.

$$\int_{-2}^{2} 5x^2 e^{0.5x}\, dx = \left[10x^2 e^{0.5x}\right]_{-2}^{2} - \int_{-2}^{2} 20xe^{0.5x}\, dx \qquad (1)$$

To solve $\int_{-2}^{2} 20xe^{0.5x}\, dx$, integration by parts is used where $u = 20x$ and $\frac{dv}{dt} = e^{0.5x}$, and $\frac{du}{dt} = 20$ and $v = 2e^{0.5x}$.

$$\int_{-2}^{2} 20xe^{0.5x}\, dx = \left[40xe^{0.5x}\right]_{-2}^{2} - \int_{-2}^{2} 40e^{0.5x}\, dx \qquad (2)$$

Substituting (2) into (1) gives:

$$\int_{-2}^{2} 5x^2 e^{0.5x}\, dx = \left[10x^2 e^{0.5x} - 40xe^{0.5x}\right]_{-2}^{2} + \int_{-2}^{2} 40e^{0.5x}\, dx$$

$$\int_{-2}^{2} 5x^2 e^{0.5x}\, dx = \left[10x^2 e^{0.5x} - 40xe^{0.5x} + 80e^{0.5x}\right]_{-2}^{2}$$

$$= 40e^1 - 80e^1 + 80e^1 - 40e^{-1} - 80e^{-1} - 80e^{-1}$$

$$= 40e - 200e^{-1}$$

*[7 marks]*
*1 mark for identifying the parts of the first integration.*
*1 mark for rewriting the first integral in parts.*
*1 mark for identifying the parts of the second integration.*
*1 mark for rewriting the second integral in parts.*
*2 marks for solving the substituted integration at key steps.*
*1 mark for providing the correct solution.*

**QUESTION 15 (7 marks)**

Finding $\frac{d\text{SA}}{dt}$ when $r = 5$ cm, where $\frac{d\text{SA}}{dt} = \frac{d\text{SA}}{dr} \cdot \frac{dr}{dt}$, gives:

$$\frac{dV}{dt} = 20 \text{ cm}^3 \text{ s}^{-1} \ (r = 3 \text{ cm, where } t = 0)$$

$$dV = 20dt$$

$$V = 20t + c$$

At $t = 0$, $r = 5$ cm. Therefore:

$$V = \frac{4}{3}\pi r^3$$
$$= 36\pi$$

Thus, $V = 20t + 36\pi$.

At $V = \frac{4}{3}\pi r^3$:

$$\frac{4}{3}\pi r^3 = 20t + 36\pi$$

$$20t = \frac{4}{3}\pi r - 36\pi$$

$$t = \frac{\frac{4}{3}\pi r^3 - 36\pi}{20}$$

$$= \frac{\pi r^3 - 27\pi}{15}$$

$$\frac{dt}{dr} = \frac{3\pi r^2}{15}$$

$$= \frac{\pi r^2}{5}$$

$$\frac{dr}{dt} = \frac{5}{\pi r^2}$$

and $\text{SA} = 4\pi r^2$

$$\frac{d_{\text{SA}}}{dr} = 8\pi r$$

$$\frac{d_{\text{SA}}}{dt} = \frac{d_{\text{SA}}}{dr} \cdot \frac{dr}{dt}$$

$$= 8\pi r \times \frac{5}{\pi r^2}$$

$$= \frac{40}{r}$$

At $r = 5$ cm:

$$\frac{d_{SA}}{dt} = \frac{40}{5}$$
$$= 8$$

When the radius of the balloon is 5 cm, the surface area will be increasing at a rate of 8 $\text{cm}^2\ \text{s}^{-1}$.

*[7 marks]*

*1 mark identifying the derivative chain* $\frac{d_{SA}}{dt} = \frac{d_{SA}}{dr} \cdot \frac{dr}{dt}$.

*1 mark for solving for V, including the constant of integration.*

*1 mark for solving for t.*

*1 mark for subsequently solving for* $\frac{dr}{dt}$.

*1 mark for solving for* $\frac{d_{SA}}{dr}$.

*1 mark for substituting* $\frac{d_{SA}}{dr}$ *into* $\frac{d_{SA}}{dt} = \frac{d_{SA}}{dr} \cdot \frac{dr}{dt}$.

*1 mark for providing the correct solution.*

**QUESTION 16 (8 marks)**

$P(z)$ will have four roots. If there are only three unique roots, two of the roots must be at $z = 6$. This leaves two roots that are complex conjugates: $z = p + iq$ and $z = p - iq$.

$P(z)$ can be factorised as $P(z) = (z-6)^2(z-p-iq)(z-p+iq)$.

Expanding the factorisation gives:

$$P(z) = (z^2 - 12z + 36)(z^2 - 2pz + p^2 + q^2)$$
$$= z^4 - (12+2p)z^3 + (p^2 + q^2 + 24p + 36)z^2 + (-12p^2 - 12q^2 - 72p)z + (36p^2 + 36q^2)$$

Equating coefficients to the original polynomial gives:

$b = -12 - 2p$ (1)

$c = p^2 + q^2 + 24p + 36$ (2)

$-96 = 12p^2 - 12q^2 - 72p$ (3)

$72 = 36p^2 + 36q^2$ (4)

From (4), it can be shown that $p^2 + q^2 = 2$. (5)

Substituting (5) into (3) gives:

$$-96 = -12(p^2 + q^2) - 72p$$
$$-96 = -24 + 72p$$
$$-96 + 24 = 72p$$
$$72 = 72p$$
$$p = 1$$

Substituting $p = 1$ into (5) gives:

$$1 + q^2 = 2$$
$$q^2 = 1$$
$$q = +1, -1$$

There are two solutions, which is reasonable as this gives the two complex conjugate roots as $z = 1 + i$ and $z = 1 - i$.

Solving for $b$ by substituting $p = 1$ into (1) gives:

$$b = -12 - 2$$
$$= -14$$

Solving for $c$ by substituting $p^2 + q^2 = 2$ and $p = 1$ into (2) gives:

$$c = 2 + 24 + 36$$
$$= 62$$

Thus, $P(z) = z^4 - 14z^3 + 62z^2 - 96z + 72$.

*[8 marks]*
*1 mark for identifying the nature of the roots (real/complex).*
*1 mark for writing P(z) as linear factors.*
*1 mark for expanding the linear factors.*
*1 mark for equating the coefficients.*
*1 mark for calculating p (real component of complex roots).*
*1 mark for calculating q (imaginary component of complex roots).*
*1 mark for calculating the coefficients b and c.*
*1 mark for writing P(z) as a polynomial with calculated coefficients.*

**QUESTION 17 (8 marks)**

For $n = 1$, $3^1 + 6^1 + 9^1 = 18$, which is divisible by 9.

Assume $3^k + 6^k + 9^k = 9A$, $k, A \in \mathbb{N}$.

For $n = k + 1$:

$$\begin{aligned}
3^{k+1} + 6^{k+1} + 9^{k+1} &= 3 \times 3^k + 6 \times 6^k + 9 \times 9^k \\
&= 3\left(3^k + 2 \times 6^k + 3 \times 9^k\right) \\
&= 3\left(3^k + 6^k + 9^k + 6^k + 2 \times 9^k\right) \\
&= 3\left(9A + 6^k + 2 \times 9^k\right) \\
&= 3 \times 9A + 3 \times 6^k + 6 \times 9^k \\
&= 3 \times 9A + 3 \times 6 \times 6^{k-1} + 6 \times 9 \times 9^{k-1} \\
&= 9\left(3A + 2 \times 6^{k-1} + 6 \times 9^{k-1}\right)
\end{aligned}$$

As each term in the brackets is an integer for $k \geq 1$, then the expression is a multiple of 9 for $k \geq 1$.

Since it is true for $n = 1$ and $n = k + 1$, it is true for $n = 2, 3, 4$ if it is also true for $n = k$.

*[8 marks]*

*1 mark for proving the proposition is true at n = 1.*

*1 mark for stating the assumption that the proposition is true at n = k and equal to 9A.*

*1 mark for identifying the proposition at n = k + 1.*

*1 mark for simplifying indices to match the assumption statement.*

*1 mark for using the substituted assumption statement.*

*1 mark for rearranging the remaining terms to fully factorise the expression by 9.*

*1 mark for providing a final statement supporting proof by induction.*

*1 mark for showing logical organisation, communicating key steps (to at least the rearrangement of proof step in an attempt to factorise the expression).*

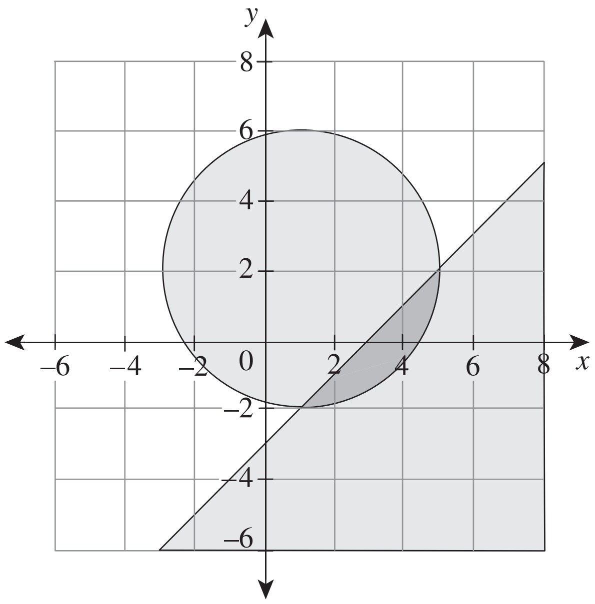
**QUESTION 18 (8 marks)**

$$|x+iy-2| \ge |x-3+i(y+1)|$$
$$\sqrt{(x-2)^2+y^2} \ge \sqrt{(x-3)^2+(y+1)^2}$$
$$(x-2)^2+y^2 \ge (x-3)^2+(y+1)^2$$
$$x^2-4x+4+y^2 \ge x^2-6x+9+y^2+2y+1$$
$$y \le x-3$$

This is a straight-line inequality with a slope of 1 and a $y$-intercept of –3, with the area below the line shaded.

$$|z-1-2i| \le 4$$
$$|x-1+i(y-2)| \le 4$$
$$\sqrt{(x-1)^2+(y-2)^2} \le 4$$
$$(x-1)^2+(y-2)^2 \le 16$$

This is a circle inequality centred on (1, 2) with a radius of 4, with the area inside the circle shaded.



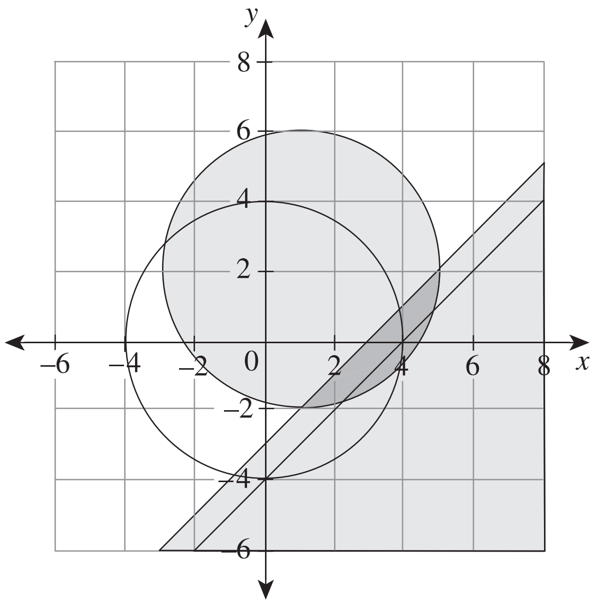

The points of intersection between the equations need to be found to calculate the intergral between the inequalities.

Substituting $y=x-3$ into $(x-1)^2+(y-2)^2=16$ gives:

$$(x-2)^2+(x-3-2)^2=16$$
$$x^2-2x+1+x^2-10+25=16$$
$$2x^2-12x+10=0$$
$$x^2-6x+5=0$$
$$(x-1)(x-5)=0$$

The points of intersection are at $x = 1$, $x = 5$.

As the area of intersection crosses the $x$-axis, the physical area can be calculated by translating the coordinates by (–1, –2) to centre the circle on the origin. This results in the area being totally below the $x$-axis and the equations of the inequalities being $y \le x-4$ and $y^2+x^2 \le 16$.




$\left(\dfrac{\pi r^2}{4}\right)$ less the area of the triangle formed between $y = x - 4$ and the $x$- and $y$-axes $\left(\dfrac{1}{2}\text{base} \times \text{height}\right)$,

or $\dfrac{\pi 4^2}{4} - \dfrac{1}{2} 4 \times 4 = 4(\pi - 2)$.

*[8 marks]*
*1 mark for finding the magnitude of each side of the first inequality.*
*1 mark for describing the characteristics of the first inequality.*
*1 mark for finding the magnitude of each side of the second inequality.*
*1 mark for describing the characteristics of the second inequality.*
*1 mark for sketching the two inequalities.*
*1 mark for finding the points of intersection of the two inequalities.*
*1 mark for transforming the inequalities to simplify the area calculation.*
*1 mark for calculating the area of the intersection.*
*Note: There are a number of acceptable approaches that can be used to solve this question. The area between the inequalities can also be calculated by inspection as the difference between the quarter circle in the fourth quadrant.*

**QUESTION 19 (3 marks)**

By inspection, the magnitude of the impedance is at a minimum when the imaginary component of the impedance is equal to zero; that is, $2\pi fL - \frac{1}{2\pi fC} = 0$.

Rearranging this for *f*, gives $f = \frac{1}{2\pi\sqrt{LC}}$.

In the case of the circuit shown:

$$f = \frac{1}{2\pi\sqrt{10\times10^{-3}\times10\times10^{-6}}}$$

$$= \frac{1}{2\pi\sqrt{100\times10^{-9}}}$$

*[3 marks]*

*1 mark for equating the imaginary component to zero.*

*1 mark for rearranging for f.*

*1 mark for calculating f.*